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An Interesting Duality in Geometry

Michael de Villiers, Mathematics Education
University of Durban-Westville
profmd@mweb.co.za
http://mzone.mweb.co.za/residents/profmd/homepage.html

*This paper will briefly discuss an interesting duality between **sides** and **angles** in Euclidean geometry. It will be presented in relation to the concepts angle and perpendicular bisector, incentre and circumcentre, and the quadrilaterals.*

*Hierdie referaat sal kortliks 'n interessante dualiteit tussen **sye** en **hoeke** in Euklidiese meetkunde bespreek. Dit sal aangebied word met betrekking tot die konsepte hoekhalveerlyne en middelloodlyne, insenter en omsenter, en die vierhoeke.*

Duality is a special kind of symmetry. In everyday language, a common duality exists between antonyms such as hot and cold, tall and short, love and hatred, male and female, etc. Basically, the one concept is defined by and understood in terms of the other, and together they form a whole which complement and enrich each other.

In mathematics, two theorems or configurations are called *dual* if the one may be obtained from the other by replacing each concept and operator by its dual concept or operator. Perhaps surprisingly, such dualities appear in many parts of mathematics, for example, projective geometry, Boolean algebra, Platonic solids, tessellations, graph theory, trigonometry, etc.

Interestingly, there exists a similar, although limited duality between the concepts *angle* (vertex or point) and *side* (line segment) within Euclidean plane geometry which we will briefly explore here. (A more extensive exploration is given in De Villiers (1996)). For example the duality between the concepts *"angle bisector"* and *"perpendicular bisector"* can be formulated as follows:

An *angle bisector* is the locus of all the points equidistant from the two *sides* of an *angle* (see Figure 1a).

A *perpendicular bisector* is the locus of all the points equidistant from the two *endpoints* of a *line segment* (side) (see Figure 1b).

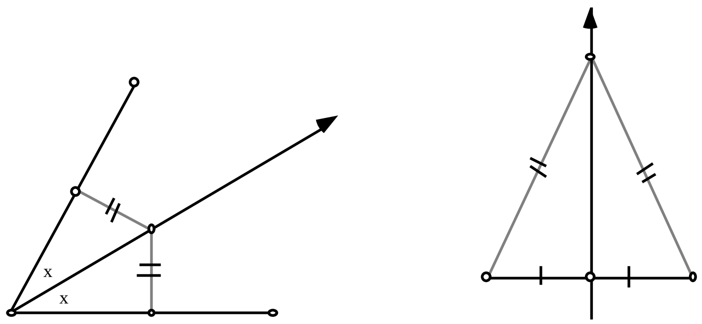

Figure 1

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The following two theorems involving these concepts are therefore also dual:

The *angle bisectors* of the *angles* of any triangle are concurrent at its *incentre* (the centre of the inscribed circle).

The *perpendicular bisectors* of the *sides* of any triangle are concurrent at its *circumcentre* (the centre of the circumscribed circle).

In fact, these two theorems can furthermore be generalised to any polygon as follows:

The *angle bisectors* of any *circum polygon* (a polygon circumscribed around a circle) are concurrent at the *incentre* of the polygon (e.g. see Figure 2a which shows a circum quad).

The *perpendicular bisectors* of any *cyclic polygon* are concurrent at the *circumcentre* of the polygon (e.g. see Figure 2b which shows a cyclic quad).

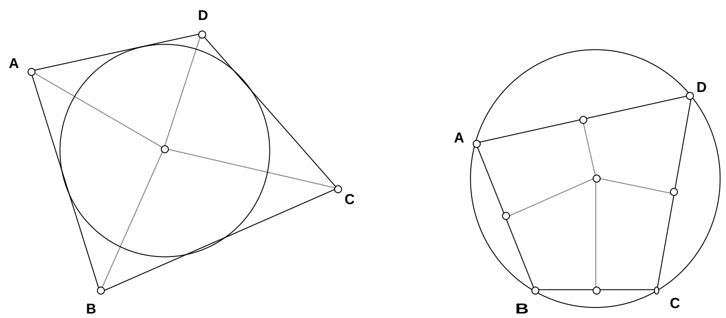

Figure 2

The general proofs are really quite simple. For example, for the first result we have that the incentre is equidistant from all the sides (radii of circle are perpendicular to sides). But each angle bisector is the locus of all points equidistant from its two adjacent sides. Therefore each angle bisector must pass through the incentre. Conversely, one should also note that this is a very useful condition for a polygon to be circumscribed around a circle. For example, for a polygon to have an incircle it must have a point which is equidistant from all the sides. Therefore, the angle bisectors must meet in a single point, i.e. be concurrent.

The second result can similarly be proved, and nicely illustrates the duality under discussion. The circumcentre is equidistant from all the vertices (radii are equal), but each perpendicular bisector is the locus of all the points equidistant from the endpoints (vertices) of each side. Therefore each perpendicular bisector must pass through the circumcentre. Conversely, one should note that this is a very useful condition for any polygon to be inscribed in a circle (be cyclic). For example, for any polygon to have a circum circle it must have a point which is equidistant from all the vertices. Therefore, the perpendicular bisectors must meet in a single point, i.e. be concurrent.

This duality between angle and side is nicely reflected in the different types of quadrilaterals as shown below in tabular form. For example, the rectangles and rhombi, isosceles trapezia and kites, and cyclic and circum quads are each other's duals. On the other hand, the squares and parallelograms are their own duals; in other words, *self-dual.*

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Square

All *angles* equal	All *sides* equal
Circumscribed circle (*cyclic*)	Inscribed circle (*circum* quad)
An axis of symmetry through each pair of opposite *sides*	An axis of symmetry through each pair of opposite *angles*

Rectangle	**Rhombus**
All *angles* equal	All *sides* equal
Circumscribed circle (*cyclic*)	Inscribed circle (*circum* quad)
An axis of symmetry through each pair of opposite *sides*	An axis of symmetry through each pair of opposite *angles*

Isosceles trapezium	**Kite**
Two pairs of equal adjacent *angles*	Two pairs of equal adjacent *sides*
One pair of equal opposite *sides*	One pair of equal opposite *angles*
Circumscribed circle (*cyclic*)	Inscribed circle (*circum* quad)
An axis of symmetry through one pair of opposite *sides*	An axis of symmetry through one pair of opposite *angles*

Cyclic quad	**Circum quad**
Circumscribed circle (*cyclic*)	Inscribed circle (*circum*)
Perpendicular bisectors of the *sides* are concurrent at the *circumcentre*	*Angle* bisectors of the *angles* are concurrent at the *incentre*
The sums of the two pairs of opposite *angles* are equal (e.g. $\angle A + \angle C = \angle B + \angle D$) (Fig 2b)	The sums of the two pairs of opposite *sides* are equal (e.g. AB + CD = BC + AD) (Fig 2a)

Parallelogram

Equal opposite *angles*	Equal opposite *sides*

This duality can be nicely displayed in the classification scheme in Figure 3 where reflection in the vertical line of symmetry gives the dual of a particular quadrilateral. (It is furthermore left to the reader to verify that the sums of the two pairs of opposite sides of a circum quad are equal).

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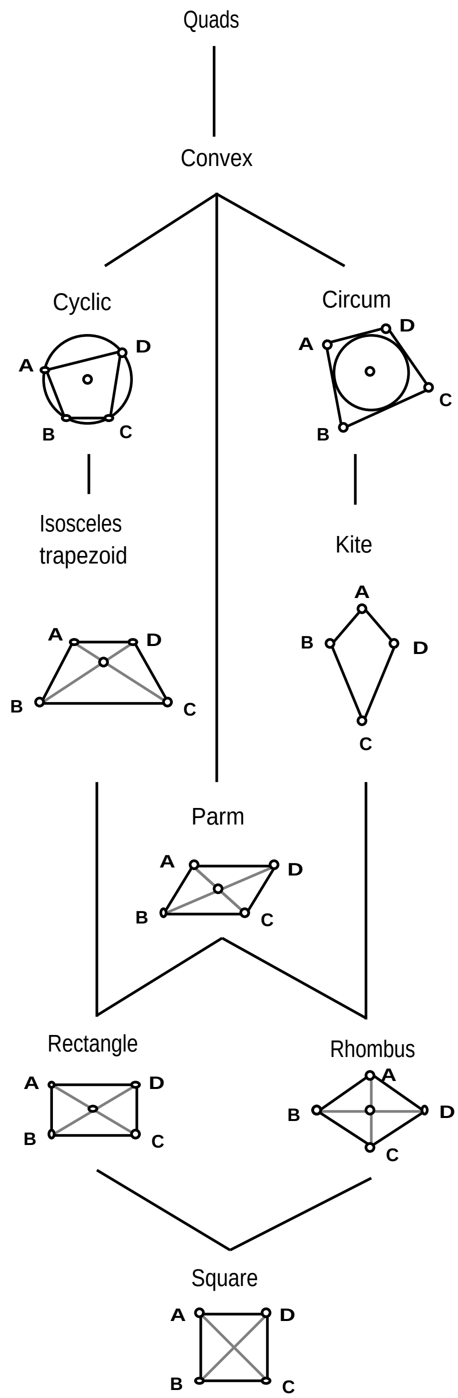

Figure 3

Many beautiful theorems in geometry display this duality. One such example is the following from De Villiers (1996).

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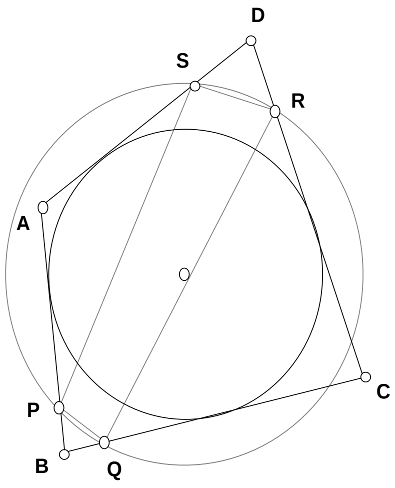

Figure 4

Theorem 1

Consider a convex circum quad as shown in Figure 4 with side lengths of AB=*a*, BC=*b*, CD=*c* and DA=*d*. Select *any* point P on AB. Take Q on BC so that BQ = PB, R on CD so that CR = QC and S on AD so that DS = RD. Then we have the surprising result that AS = AP and PQRS is a cyclic quadrilateral. (Note that this result is a generalization of the result that the four points where the incircle touches the sides of a circum quad are concyclic).

Investigate this theorem dynamically with the use of Sketchpad!

1. If you do not have Sketchpad, first download a **FREE DEMO** of it from: http://www.keypress.com/sketchpad/sketchdemo.html
2. Then download the following to dynamically investigate this theorem: http://mzone.mweb.co.za/residents/profmd/amesa96a.zip

Challenge: Can you prove this theorem?

Stuck? Or want to compare your proof?

Download the proof from:

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Theorem 2

Construct any *angle divider* AQ of $\angle A$ of a convex *cyclic quad* ABCD. Now construct the angle divider BS of $\angle B$ so that $\angle PBA = \angle PAB$, the angle divider CR of $\angle C$ so that $\angle SCB = \angle SBC$ and the angle divider DQ of $\angle D$ so that $\angle RDC = \angle RCD$ (see Figure 5). Then $\angle QDA = \angle QAD$ and PQRS is a *circum quad*.

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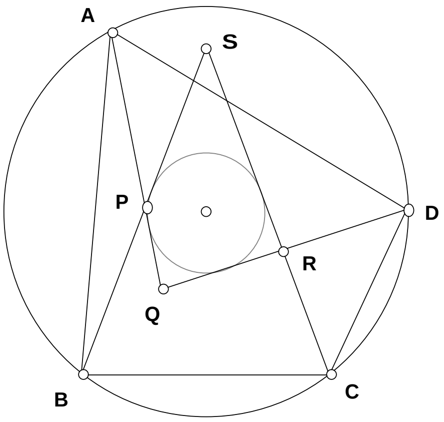

Figure 5

Investigate this theorem dynamically with the use of Sketchpad!

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2. Then download the following to dynamically investigate this theorem:
   http://mzone.mweb.co.za/residents/profmd/amesa96b.zip

Challenge: Can you prove this theorem?

Stuck? Or want to compare your proof?

Download the proof from:

http://mzone.mweb.co.za/residents/profmd/amesa96b.pdf

Exercise

1. Can you generalize the two dual results given in Theorems 1 and 2 to hexagons, octagons, etc?
2. Can you find analogous results related to triangles for the two dual results given in Theorems 1 and 2? If so, can you generalize?
   (Hint: See http://mzone.mweb.co.za/residents/profmd/sharp.pdf)

Reference

De Villiers, M. (1996). *Some Adventures in Euclidean Geometry*. Durban: University of Durban-Westville.

(Ordering information: http://mzone.mweb.co.za/residents/profmd/homepage2.html)