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# A generalization of the Fermat-Torricelli point

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It is possible to generalize the result involving the Fermat-Torricelli point to similar triangles or similar isosceles triangles on the sides as follows:
"If similar triangles *DBA*, *CBE* and *CFA* are constructed outwardly on the sides of any *ABC*, then *DC*, *EA* and *FB* are concurrent ".
"If similar isosceles triangles *DBA*, *ECB* and *FAC* are constructed outwardly on the sides of any ABC so that $\angle DAB = \angle DBA$, then *DC*, *EA* and *FB* are concurrent ".

With regard to the first case note:
(a) if $a = \frac{1}{2}(\angle B + \angle C)$, $b = \frac{1}{2}(\angle A + \angle C)$ and $c = \frac{1}{2}(\angle A + \angle B)$, then *DC*, *EA* and *FB* are the angle bisectors of triangle *ABC*
(b) if triangles *DBA*, *CBE* and *CFA* are congruent, then *DC*, *EA* and *BF* are the altitudes of triangle *ABC*.
Also note with regard to the second case above:

(c) if $a = 0^\circ$, then *DC*, *EA* and *FB* are the medians of triangle *ABC*.

These two results are therefore nice generalizations of the familiar concurrencies of the angle bisectors, altitudes and medians.

However, a further unifying generalization is possible, for example:
"If triangles *DBA*, *ECB* and *FAC* are constructed outwardly on the sides of any $\Delta ABC$ so that $\angle DAB = \angle CAF$, $\angle DBA = \angle CBE$ and $\angle ECB = \angle ACF$ then *DC*, *EA* and *FB* are concurrent."

To prove this result it is first necessary to prove the following lemma.

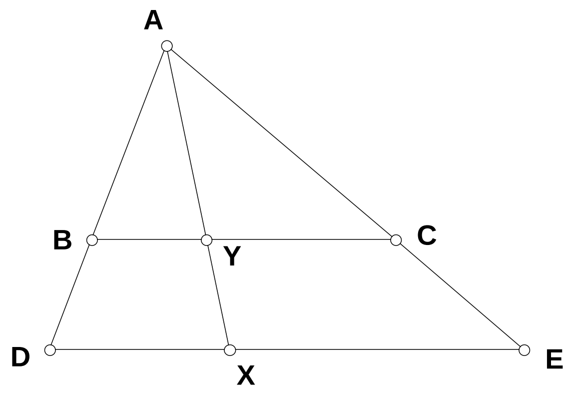


**Figure 3.11**

**Lemma**

Triangle *ABC* is given. Extend *AB* and *AC* to *D* and *E* respectively so that *DE//BC*. Choose any point *Y* on *BC* and extend *AY* to *X* on *DE* (see Figure 3.11). Then *BY/YC* = *DX/XE*.

**Proof**

Since triangles *ABY* and *ADX* are similar we have *BY/YA* = *DX/XA* and therefore *BY* = (*YA/XA*).*DX* ... (1). Similarly from the similarity of triangles *ACY* and *AEX* we have *CY* = (*YA/XA*).*EX* ... (2). Dividing (1) by (2), gives *BY/YC* = *DX/XE*, the desired result.

**Proof of the Fermat generalization**

Assume that the lines we want to prove concurrent intersect *BC*, *CA* and *AB* respectively at *X*, *Y* and *Z*. Extend *AB* to *G* and *AC* to *H* so that *GEH*//*BC* (see Figure 3.12). Label *BE*, *EC*, *CF*, *FA*, *AD* and *DB* respectively as $s_1, s_2, s_3, s_4, s_5$ and $s_6$. Then $\angle BGE = \angle ABC$ and $\angle BEG = b$.

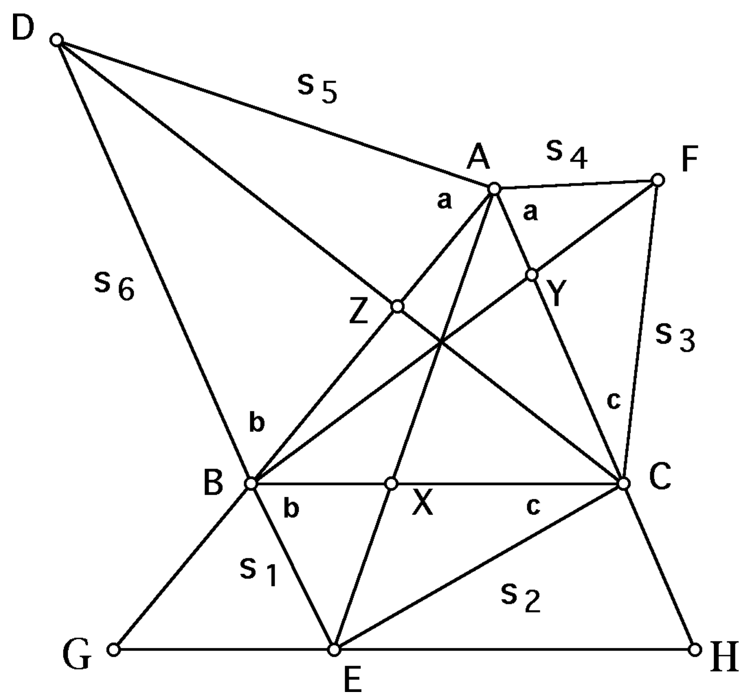


**Figure 3.12**

According to the sine rule:

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$$\frac{GE}{\sin(\angle GBE)} = \frac{s_1}{\sin(\angle ABC)}$$

$$\frac{GE}{\sin(b + \angle ABC)} = \frac{s_1}{\sin(\angle ABC)}$$

$$GE = \frac{s_1 \sin(b + \angle ABC)}{\sin(\angle ABC)}$$

Similarly we obtain

$$EH = \frac{s_2 \sin(c + \angle ACB)}{\sin(\angle ACB)}.$$

According to the preceding Lemma therefore

$$\frac{BX}{XC} = \frac{GE}{EH} = \frac{s_1 \sin(b + \angle ABC)}{\sin(\angle ABC)} \cdot \frac{\sin(\angle ACB)}{s_2 \sin(c + \angle ACB)}.$$

In the same way we have

$$\frac{CY}{YA} = \frac{s_3 \sin(c + \angle ACB)}{\sin(\angle ACB)} \cdot \frac{\sin(\angle CAB)}{s_4 \sin(a + \angle CAB)}$$

$$\frac{AZ}{ZB} = \frac{s_5 \sin(a + \angle CAB)}{\sin(\angle CAB)} \cdot \frac{\sin(\angle ABC)}{s_6 \sin(b + \angle ABC)}$$

Therefore, $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{s_1}{s_2} \cdot \frac{s_3}{s_4} \cdot \frac{s_5}{s_6}$ ... (3)

Applying the sine rule to triangles *ECB*, *FAC* and *DBA* we obtain

$$\frac{s_1}{s_2} = \frac{\sin(c)}{\sin(b)}; \frac{s_3}{s_4} = \frac{\sin(a)}{\sin(c)}; \frac{s_5}{s_6} = \frac{\sin(b)}{\sin(a)}$$

By substitution into (3) therefore $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1$ so that $AX$, $BY$ and $CZ$ are concurrent according to the converse of Ceva's theorem (see Problem 3). But then *EA*, *FB* and *DC* are also concurrent.

The author has since learned of an earlier and rather simpler proof of this generalization given by A. R. Pargeter in *The Mathematical Gazette*, Vol. 47, no. 364, pp. 218-219, and an even earlier 1936 proof by N. Alliston in *The Mathematical Snack Bar* by W. Hoffer, pp. 13-14. (These earlier proofs, however, do not mention the interesting associated result given in Equation (3) above, namely, that the product of the given side ratios of the constructed triangles (or their reciprocals) is always equal to 1).