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A Dual to Kosnita's theorem

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In the September 1995 issue of the *Mathematics and Informatics Quarterly*, the following interesting result was mentioned without proof in the column on Forgotten Theorems:

> **Kosnita's Theorem**. The lines joining the vertices *A*, *B*, and *C* of a given triangle *ABC* with the circumcenters of the triangles *BCO*, *CAO*, and *ABO* (*0* is the circumcenter of Δ *ABC*), respectively, are concurrent.

On the basis of an often observed (but not generally true) duality between circumcenters and incenters (eg. see De Villiers, 1996), I immediately conjectured that the following dual to Kosnita's theorem might be true, namely:

> **Kosnita Dual**. The lines joining the vertices *A*, *B*, and *C* of a given triangle *ABC* with the incenters of the triangles *BCO*, *CAO*, and *ABO* (*0* is the incenter of Δ *ABC*), respectively, are concurrent.

Investigation on the dynamic geometry program *Sketchpad* then confirmed that the conjecture was indeed true. I then found that both results could easily be proved by the following useful result for concurrency, a proof of which is given in De Villiers (1996). Interestingly, the point of concurrency for the special case with equilateral triangles on the sides is called the Fermat-Torricelli point.

> **Fermat-Torricelli Generalization**. If triangles *DBA*, *ECB* and *FAC* are constructed on the sides of any triangle *ABC* so that $\angle DAB = \angle CAF$, $\angle DBA = \angle CBE$ and $\angle ECB = \angle ACF$ then *DC*, *EA* and *FB* are concurrent (see Figure 1).

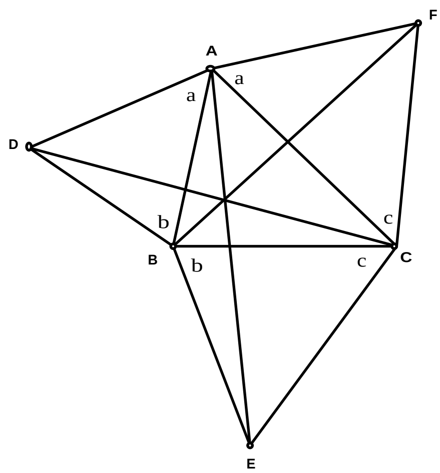

Figure 1

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It should further be pointed out that the above result is also true when the triangles are constructed inwardly. As shown in Figure 2 we clearly have for Kosnita's dual that $\angle DAB = \frac{1}{4}\angle A = \angle CAF$, $\angle DBA = \frac{1}{4}\angle B = \angle CBE$ and $\angle ECB = \frac{1}{4}\angle C = \angle ACF$, and from the above result it therefore follows that *DC*, *EA* and *FB* are concurrent.

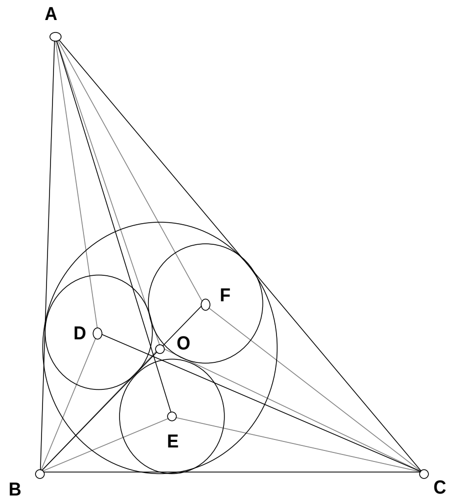

Figure 2

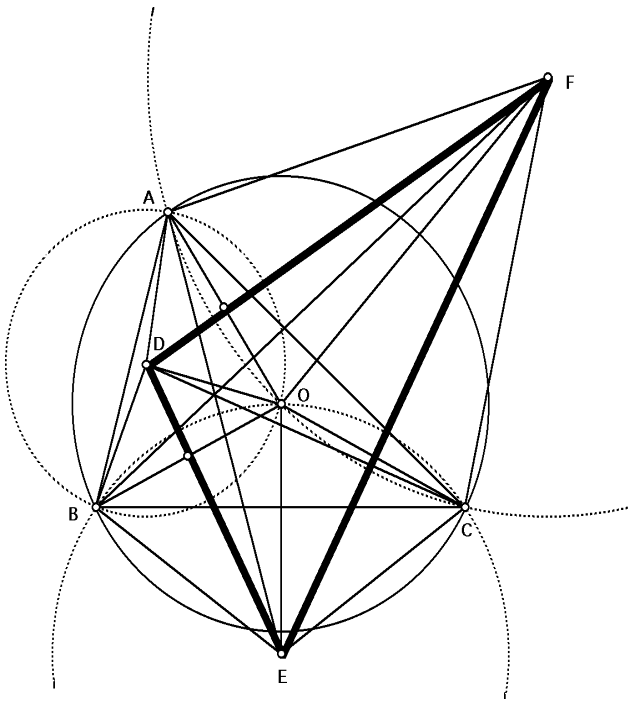

Figure 3

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Kosnita's theorem follows a little less directly from the Fermat-Torricelli generalization. In this case, the "base triangle" is triangle *DEF* with *A*, *B* and *C* the outer vertices (see Figure 3). Since *DBOA* is a kite, we have $\angle BDO = \angle ADO$. But *DBEO* and *DOFA* are also kites. Therefore, $\angle BDE = \frac{1}{2}\angle BDO$ and $\angle ADF = \frac{1}{2}\angle ADO$ from which follows that $\angle BDE = \angle ADF$. In a similar fashion can be shown that $\angle BED = \angle CEF$ and $\angle CFE = \angle AFD$. From the F-T generalization, it therefore follows that *DC*, *EA* and *FB* are concurrent.

Reference

De Villiers, M. (1996). *Some Adventures in Euclidean Geometry.* Durban: University of Durban-Westville.