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Exploring Loci on Sketchpad

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"*The fascination of mathematics is fundamentally the same as the fascination of exploration except that the discoveries are made in the realm of ideas rather than in physical space. No doubt the pleasure is greatest when an idea is clarified and isolated after a struggle, but most people have sufficient experience, if only in attempting to solve Christmas puzzles, to enable them to understand the exuberance with which Pythagoras and Archimedes are said to have celebrated their discoveries. It is not possible for our pupils to rediscover the whole of mathematics for themselves, or even those portions of it which seem to the greatest relevance today, but fortunately the pleasure seems to be experienced under guided discovery. It is important that the classroom activities should be carried on with a certain degree of expectancy; new ideas, fresh discoveries, deepened interest are just round the corner waiting to burst in at any moment.*" - Bailey et al (1974: 148)

How is new mathematics discovered or created? Where does this theorem or that theory come from? How was it arrived at? What stimulated its conjecture or development?

These are burning questions that are seldom adequately answered for our pupils and students. Traditionally, the teacher just announces a theorem like a magician pulling a rabbit from a hat, leaving pupils (subconsciously) wondering where it came from or how it could have been discovered, and therefore adding to the unsatisfactory mystification of mathematics.

The arrival of software like *Geometer's Sketchpad* provides an incredible dynamic tool for exploring geometry, and facilitating conjecturing. Basic explorations of triangles, quadrilaterals, circles, and other geometric figures are a breeze with *Sketchpad.* Using *Sketchpad* gives pupils or students the power to explore actively, without the mechanical restraints of pencil and paper, compass, and straightedge. It allows your pupils or students to dynamically transform their figures with the mouse, while preserving the geometric relationships of their constructions. They'll be able to examine an entire set of similar cases in a matter of seconds, leading them naturally to generalizations. *Sketchpad* therefore encourages a process of discovery where pupils or students first visualise and analyze a problem, and make conjectures before attempting a proof.

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Although loci are no longer part of our school syllabus, the availability of dynamic geometry software with locus tracing facilities such as *Sketchpad*, certainly makes its reconsideration a strong possibility. For example, the conics (circle, parabola, ellipse and hyperbola) can be beautifully illustrated as loci (ie. the classical Greek treatment), from which the standard equations can then later be derived. Such a historical approach would certainly be very much in line with the learning outcomes of the OBE approach. Many real world problems also involve loci which can easily be modelled with *Sketchpad*.

What follows below are three examples of exploring loci with *Sketchpad*.

Explore dynamically by downloading a FREE DEMO of SKETCHPAD from:
http://www.keypress.com/sketchpad/sketchdemo.html

Example 1

The ancient Greeks carried out intensive investigations on loci. For example, the circle was seen to be the *locus* (the set of points) such that each point on the locus was equidistant from a fixed point (the center). Similarly, the investigation of loci with the respective properties that each point on the locus is equidistant to two fixed points or to two fixed lines, respectively give a perpendicular bisector, and an angle bisector.

It is also natural to ask: What is the locus (the set of points) equidistant from a fixed point and a fixed line?

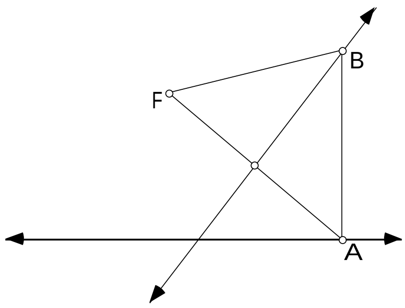

Figure 1

Consider Figure 1 which shows a fixed point F and a fixed straight line, with A an arbitrary point A on the line. All points equidistant from F and A lie on the perpendicular bisector of FA; so the desired point of the locus must lie somewhere on this perpendicular bisector. Since the distance from any point on the locus to the line is the perpendicular distance, it follows that the desired point B can be obtained by constructing the perpendicular at A to intersect the perpendicular bisector.*Sketchpad* allows us to dynamically investigate the path of this point B as A is move along the line. For example, select point B and choose Trace Locus from the Display menu. As A is dragged along the line we obtain Figure 2 which is the desired curve.

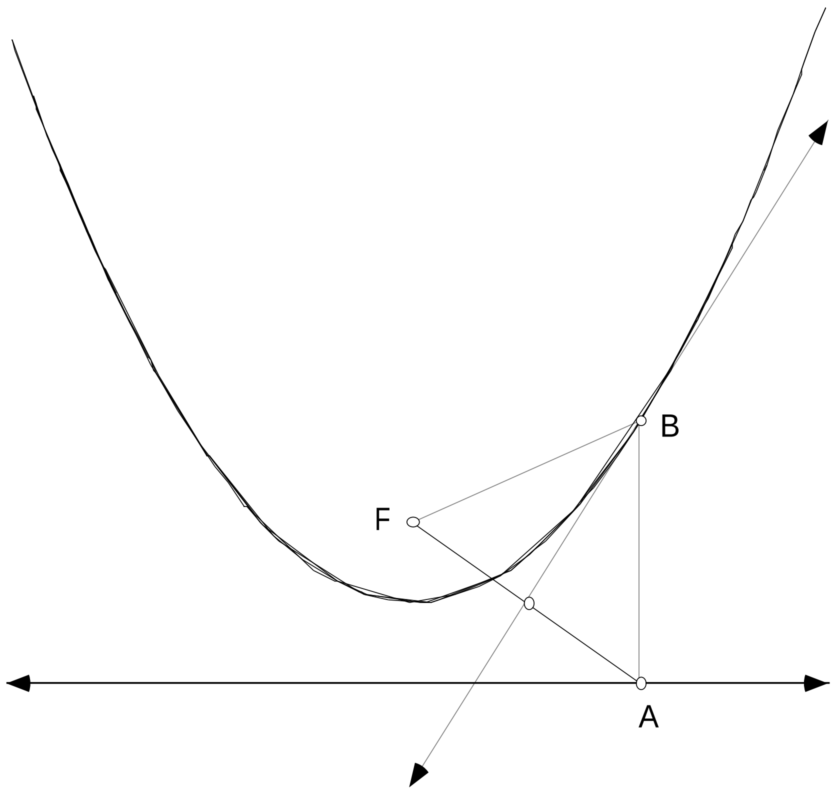

Figure 2

One of the advantages of *Sketchpad* is of course its dynamic nature and one could now also examine the effect of dragging F further away or nearer to the line. This curve is of course the well-known parabola where F is called the **focus** and the line the **directrix**. Can we determine an algebraic equation for this curve?

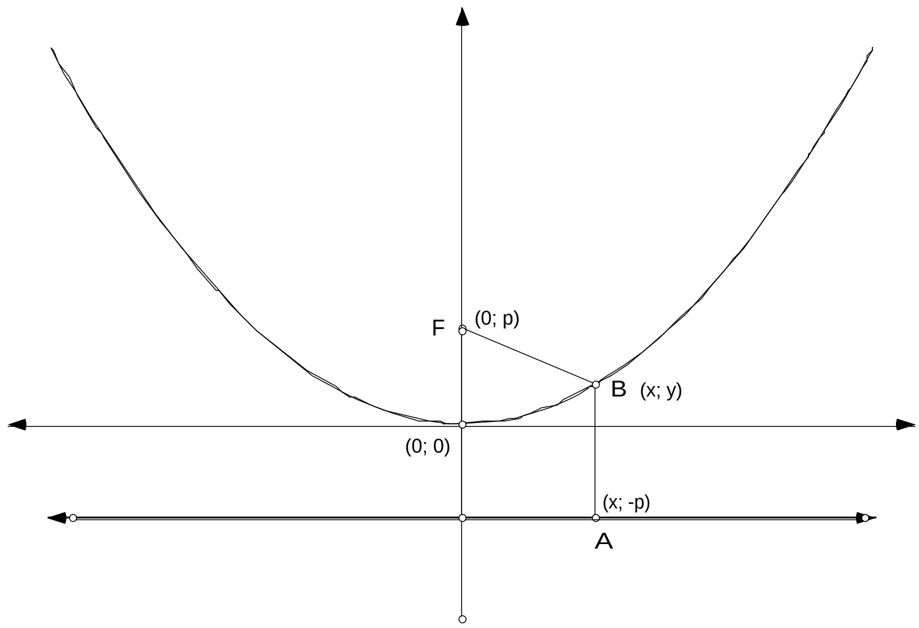

Figure 3

Suppose as shown in Figure 3 the coordinates of the vertex of the parabola are (0; 0) and that of the focus F (0; p). Since the vertex is halfway between the directrix and the focus, it follows that the coordinates of point A are (x; -p). Since FB = AB we have:

$$(x-0)^2 + (y-p)^2 = (y+p)^2$$

$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2$$

$$y = \frac{x^2}{4p}$$

This is clearly the standard equation of a parabola of the form $y = ax^2$. One of advantages of *Sketchpad* is that it encourages the asking of "*what-if?*" questions because of the ease by which such questions can be investigated. For example, what is the locus of the midpoint C of BX? As shown in Figure 3, this locus appears to be also a parabola. Can you provide an explaination why it must be a parabola, or why it is not? If a parabola, will the locus of any point on the perpendicular bisector dividing BX in a fixed ratio (internally or externally), be a parabola? These questions are however left to the reader to investigate.

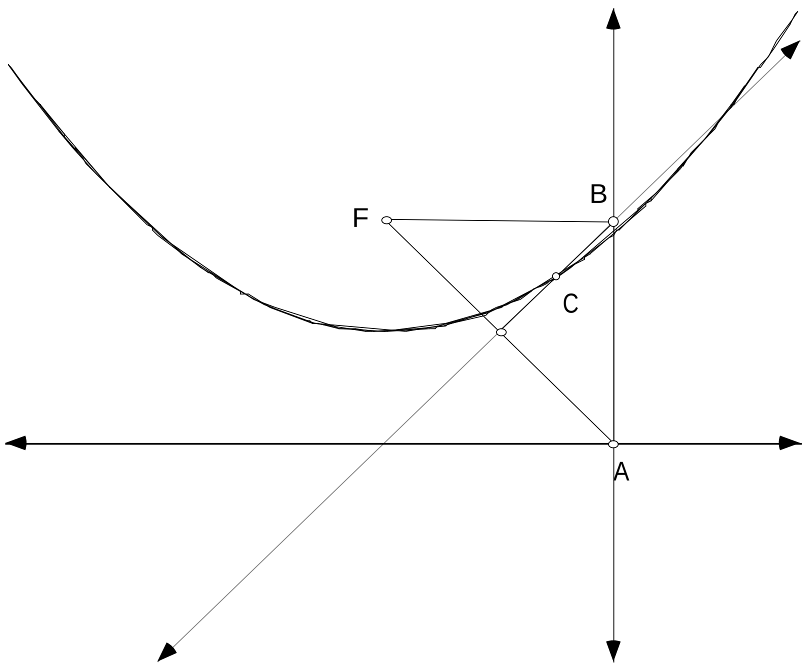

Figure 4

The hyperbola and ellipse can similarly be explored as loci with *Sketchpad* (compare Scher, 1995).

Example 2

Consider the following interesting problem, namely:

"Points M and N lie on sides AC and BC of triangle ABC with AM = BN. What is the locus of the midpoint P of MN as M and N vary?"

For example, a dynamic configuration to visually illustrate this result can easily be obtained by constructing, firstly, a line through A and choosing M as an arbitrary point on that line, and secondly, a parallelogram MABX as illustrated in Figure 5 to ensure that AM=BX. Then by constructing a circle with B as centre and BX as radius, and choosing any point N on the circle B, we ensure the equality of AM and BN. Finally, we construct a line BN to intersect line AM in C, and P as the midpoint of MN. By now choosing the *Trace locus* facility and selecting P, the locus of P is traced out as M is moved back and forth along line AC (see Figure 5). One can now also clearly see that this locus is parallel to the angle bisector of $\angle C$.

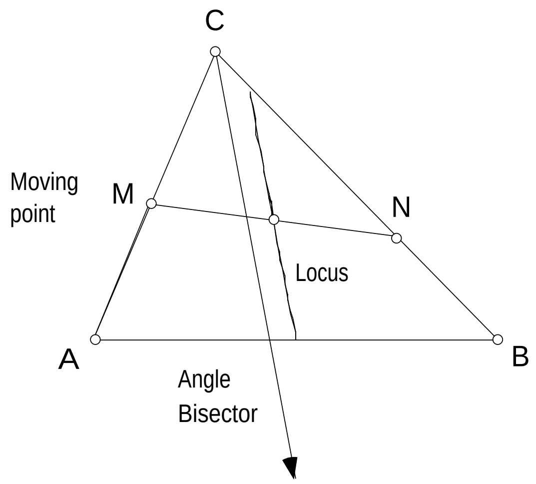

Figure 5

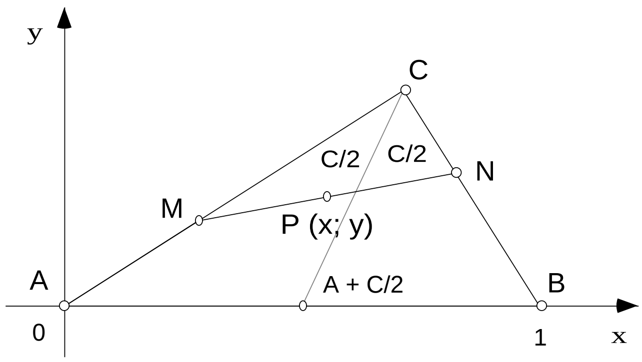

Figure 6

The result can be proved as follows. Set ΔABC in a coordinate system with A at (0, 0), B at (1, 0) and C in the first quadrant as shown in Figure 6. Suppose AM=BN=*t*. Then M has coordinates ($t\cos A$, $t\sin A$), N has coordinates ($1-t\cos B$, $t\sin B$). Since P(*x*, *y*) is the midpoint of MN, we have:

(1) $x=\frac{1}{2}(t\cos A+1-t\cos B)$

(2) $y=\frac{1}{2}(t\sin A+\sin B)$

From (2), we obtain $t=\dfrac{2y}{\sin A+\sin B}$ and substituting in (1):

$$x = \tfrac{1}{2} + y\left[\frac{\cos A - \cos B}{\sin A + \sin B}\right]$$

$$\text{or } y = \left[\frac{\sin A + \sin B}{\cos A - \cos B}\right](x - \tfrac{1}{2})$$

This shows that the locus of P is a straight line. Using two trigonometric identities (which are not in the school syllabus, but should be better known) we have:

$$\begin{aligned}\frac{\sin A + \sin B}{\cos A - \cos B} &= \frac{2\sin\frac{A+B}{2}\cos\frac{B-A}{2}}{2\cos\frac{A+B}{2}\sin\frac{B-A}{2}} \\ &= \cot\frac{B-A}{2} \\ &= \tan\left[90° - \tfrac{B-A}{2}\right] \\ &= \tan\left[90° - \tfrac{(180°-A-C)-A}{2}\right] \\ &= \tan\left[A + \tfrac{C}{2}\right]\end{aligned}$$

Now $A + \frac{C}{2}$ is the angle at which the internal bisector of angle C cuts AB (see Figure 6). Hence the straight line locus is parallel to the internal bisector of angle C.

One could now also explore some different variations on the original Sharp problem by asking a few "*what-if?*" questions which will be left to the reader for further exploration (see De Villiers, 1996).

An obvious "*what-if?*" question is: What happens if P is not the midpoint of MN, but divides MN in the ratio *p:q*, i.e. so that $MP/PN = p/q$? Will the locus of P still be a straight line parallel to the angle bisector of $\angle C$?

Another "*what-if?*" question is: What happens if AM≠BN, but in a constant ratio *r:s* to each other, i.e. $AM/BN = r/s$? Will the locus still be a straight line parallel to the angle bisector of $\angle C$?

The dynamic construction also suggests the following "*what-if?*" question (with regard to the quadrilateral AMNB): What happens to the locus of P if N is moved around the circumference of the circle with centre B and radius BN (=BX)?

Example 3

Consider the following problem:

"What is the locus of the orthocentre of a triangle ABC if B and C are fixed and A is moved along the circumference of the (fixed) circumcircle of triangle ABC?"

Investigating the problem on *Sketchpad* as shown in Figure 7, reveals that the locus is a circle congruent to the circumcircle of triangle ABC. Its proof is left as a challenge to the reader.

Hint: 1. Consult De Villiers, 1996b & Pillay, 1997 for a helpful, related result.

2. See No. 3 in http://mzone.mweb.co.za/residents/profmd/spsol00.pdf

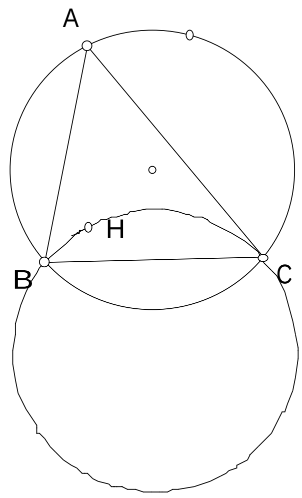

Figure 7

Notes:

(1) *Geometer's Sketchpad* and additional materials are available from *Dynamic Learning*, 8 Cameron Rd, Sarnia (Pinetown) 3615 and runs on IBM (386 upwards, 4MB RAM, Windows) or Apple Macintosh. Tel: 031-2044252 (w) or 031-7029941 (Pearl); 031-7083709 (h); E-mail: dynamiclearn@mweb.co.za

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