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From nested Miquel triangles to Miquel distances

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This article presents interesting generalizations of three well-known results related to pedal triangles and distances, and the Simson-Wallace line.

Nested Miquel triangles

The triangle whose vertices are the feet of the perpendiculars from a point *P* inside a triangle *ABC* to each of its sides *AB*, *BC* and *AC*, is called a *pedal* triangle. The pedal triangle has many interesting properties, but the following one is particularly interesting (see [1]).

Neuberg's theorem: If a sequence of nested pedal triangles from the same point *P* is constructed, then the third pedal triangle is similar to the original triangle *ABC*.

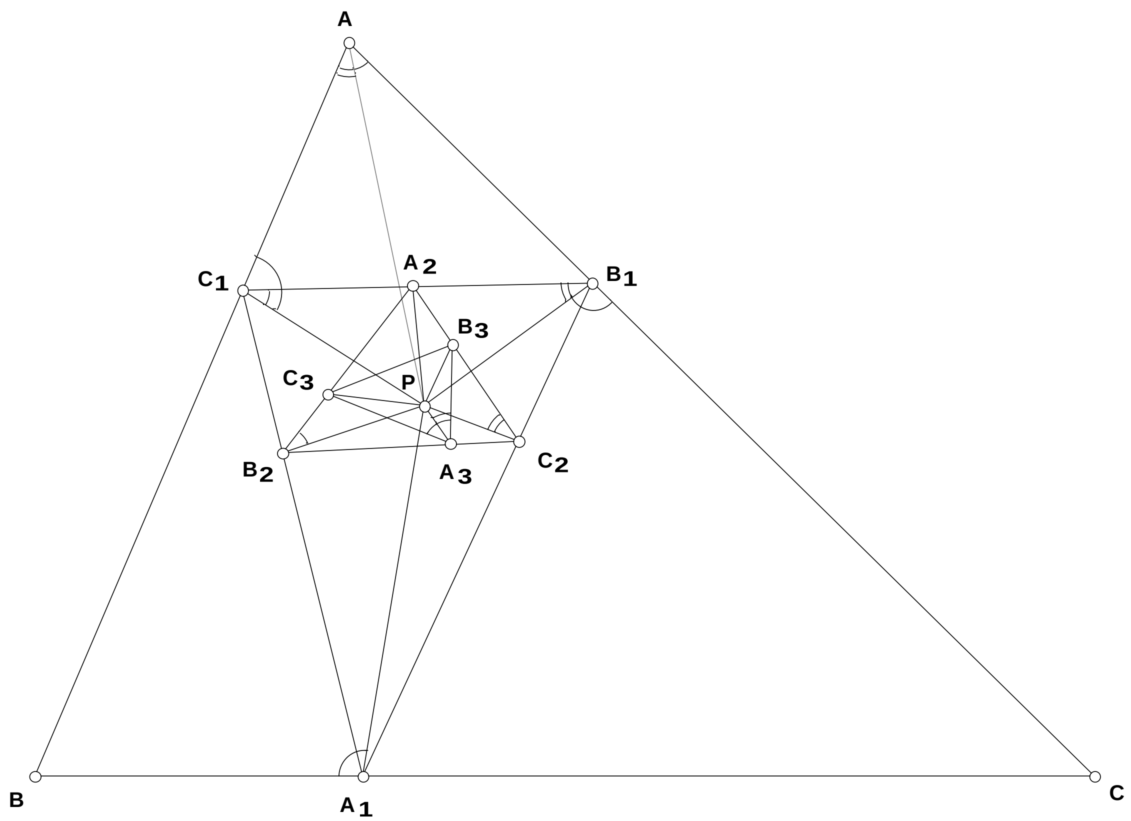

Figure 1

Neuberg's theorem can, however, be generalized by starting with a point *P* and constructing lines to the sides of a triangle *ABC* so that these lines all form equal angles with the sides as shown in Figure 1 (i.e. $\angle PA_1B = \angle PB_1C = \angle PC_1A$). Following [2], we shall call the triangle $A_1B_1C_1$ formed by these lines, a Miquel triangle. From the same point *P*, construct a second Miquel triangle in the first Miquel triangle, and then another Miquel triangle in the second one. Then the third Miquel triangle is similar to the original triangle *ABC*.

What makes this generalization even more surprising is that the subsequent Miquel triangles can be constructed entirely arbitrarily. Indeed, when I first constructed these lines on *Sketchpad* to make equal angles with the sides, I assumed that the next set of lines needed to form the same angles with the sides as for the first Miquel triangle (or that the next set of angles needed to be supplementary with the preceding ones). However, on constructing a proof and reflecting upon it, I realized that this condition was not necessary at all, and that arbitrary Miquel triangles would do. This is therefore another good example of proof as a means of *discovery* as discussed and illustrated in [3] & [4].

Proof

A proof immediately becomes apparent when one draws the segment *PA*. From the construction of the lines to the sides, we have that quadrilaterals AC_1PB_1, $A_2PC_2B_1$ and $B_3PA_3C_2$ are all cyclic. Therefore, $\angle PAC_1 = \angle PB_1C_1 = \angle PC_2A_2 = \angle PA_3B_3$. Similarly, we can show that $\angle PAB_1 = \angle PA_3C_3$. Thus, $\angle B_3A_3C_3 = \angle BAC$. In the same way, the equality of corresponding angles at *B* and B_3 or *C* and C_3 can be established. Q.E.D.

Reflection

Since the Miquel triangles can be constructed arbitrarily at each stage, I realized that this implied that they are all similar at each stage; indeed they all had to be similar to the respective pedal triangles at each stage.

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So an alternative proof would simply be to establish the similarity of all the Miquel triangles at a particular stage. This is now given below.

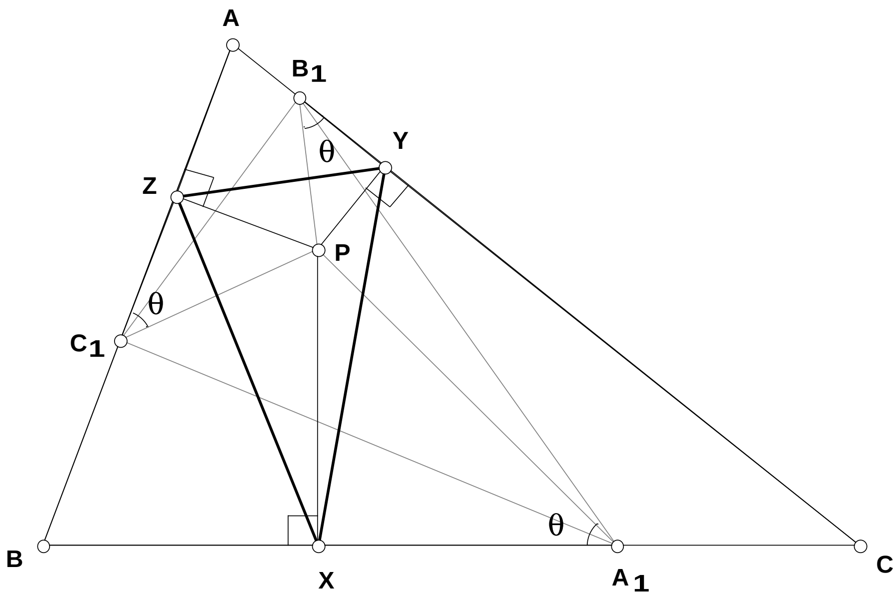

Figure 2

Alternative Proof

Consider Figure 2 containing a pedal triangle XYZ and an arbitrary Miquel triangle $A_1B_1C_1$ falling on the sides of ABC (or the sides extended if necessary). Since right triangles PXA_1, PYB_1 and PZC_1 are all similar by construction, it follows that $\angle XPA_1 = \angle YPB_1 = \angle ZPC_1$. Therefore, a spiral similarity with P as centre - i.e. a rotation of pedal triangle XYZ around P through $\angle XPA_1$, and an enlargement of it with the ratio $\frac{PA_1}{PX}$ - maps the pedal triangle XYZ onto the arbitrary Miquel triangle $A_1B_1C_1$. Similarly, all Miquel triangles at each nesting stage are directly similar to the unique pedal triangle of that stage, but since the third nested pedal triangle is similar to triangle ABC according to Neuberg's theorem, any third nested Miquel triangle would also be similar to triangle ABC. Q.E.D.

Stewart found the following interesting generalization: the nth pedal n-gon of any n-gon is similar to the original n-gon (see [5]). In exactly the same way as above, it is easy to prove that a spiral similarity would map the pedal *n*-gon at any stage onto an infinite number of similar Miquel polygons. Thus, Stewart's theorem immediately generalizes to Miquel *n*-gons.

Sum of Miquel distances

A reasonably well-known result is that the sum of the distances from any point *P* inside an equilateral triangle to each of its sides is constant. This result also generalizes to *P* outside the equilateral triangle, provided one uses directed distances. It also further generalizes to equi-sided polygons, as well as to equi-angular polygons.

If we let the lines from any point *P* all form *fixed* angles θ with the sides and call the distance from *P* to a Miquel vertex, the Miquel distance, then the sum of the Miquel distances to the sides of equi-sided (or equi-angular) polygons, is also constant.

Proof

This is immediately apparent from Figure 2. Since $PX + PY + PZ + \ldots$ is constant for any position of *P* for equi-sided (or equi-angular) polygons, it follows that $PA_1 \sin\theta + PB_1 \sin\theta + PC_1 \sin\theta + \ldots$ is constant, and therefore also $PA_1 + PB_1 + PC_1 + \ldots$. Q.E.D.

The Miquel line

If point P lies on the circumcircle of triangle *ABC* and perpendiculars from it are dropped onto the sides of *ABC*, then the pedal triangle degenerates into a straight line, the so-called Simson line.

In general, if point *P* lies on the circumcircle of triangle *ABC* and lines from it are drawn to the sides of *ABC* (suitably extended if necessary) to form *fixed* angles θ with the sides, then the Miquel triangle degenerates into a straight line, which we shall call the Miquel line.

Proof

This generalization of the Simson (or Wallace) line is apparently not well known, though it is explicitly stated and proved in [2, p. 137]. To establish the general result, it is also possible to simply generalize the proof for the Simson line given in [1, p. 40]. A slightly different proof to either of these two will now be given below.

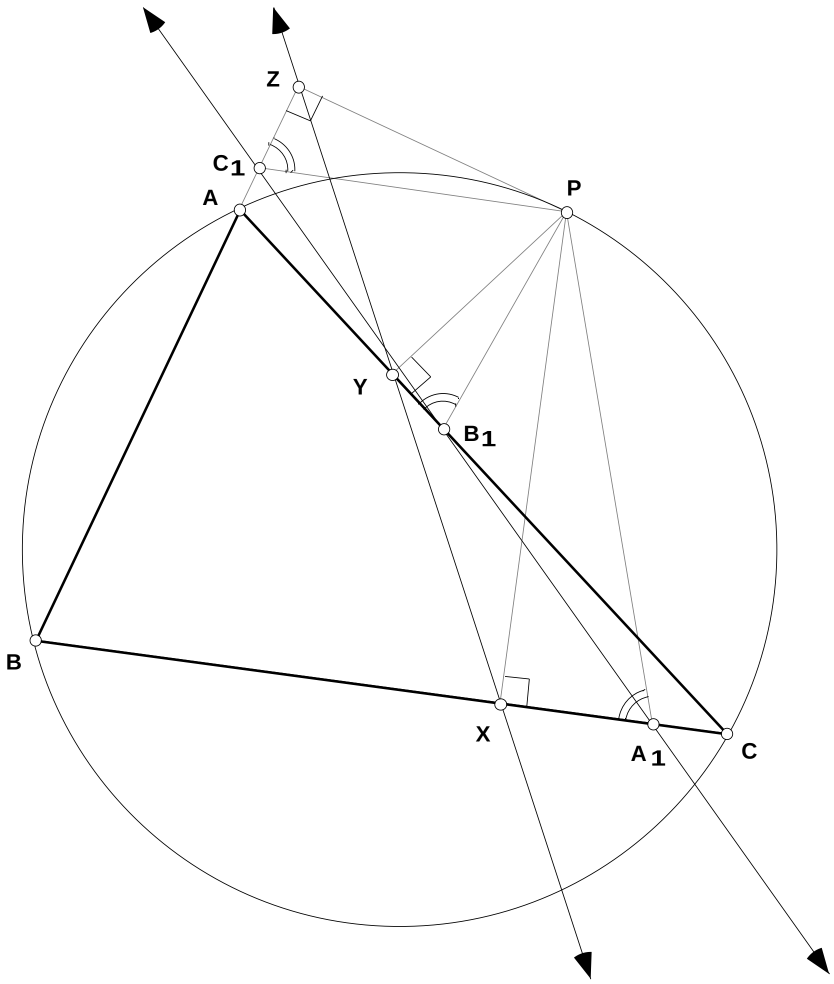

Figure 3

Consider Figure 3 with a Simson line XYZ and a Miquel triangle $A_1B_1C_1$. Exactly as in Figure 2, a spiral similarity with P as centre - i.e. a rotation of the Simson line XYZ around P through $\angle XPA_1$, and an enlargement of it with the ratio $\frac{PA_1}{PX}$ - maps the Simson line onto the Miquel triangle $A_1B_1C_1$. Therefore, $A_1B_1C_1$ is also a straight line. Q.E.D.

The Simson (Wallace) line has many interesting properties that generalize directly to the Miquel line. However, for the purpose of this article, we will only consider a few properties of Miquel lines that either follow directly from the spiral similarity or can be proved by the reader by generalizing corresponding Simson proofs in [1, pp. 43-45].

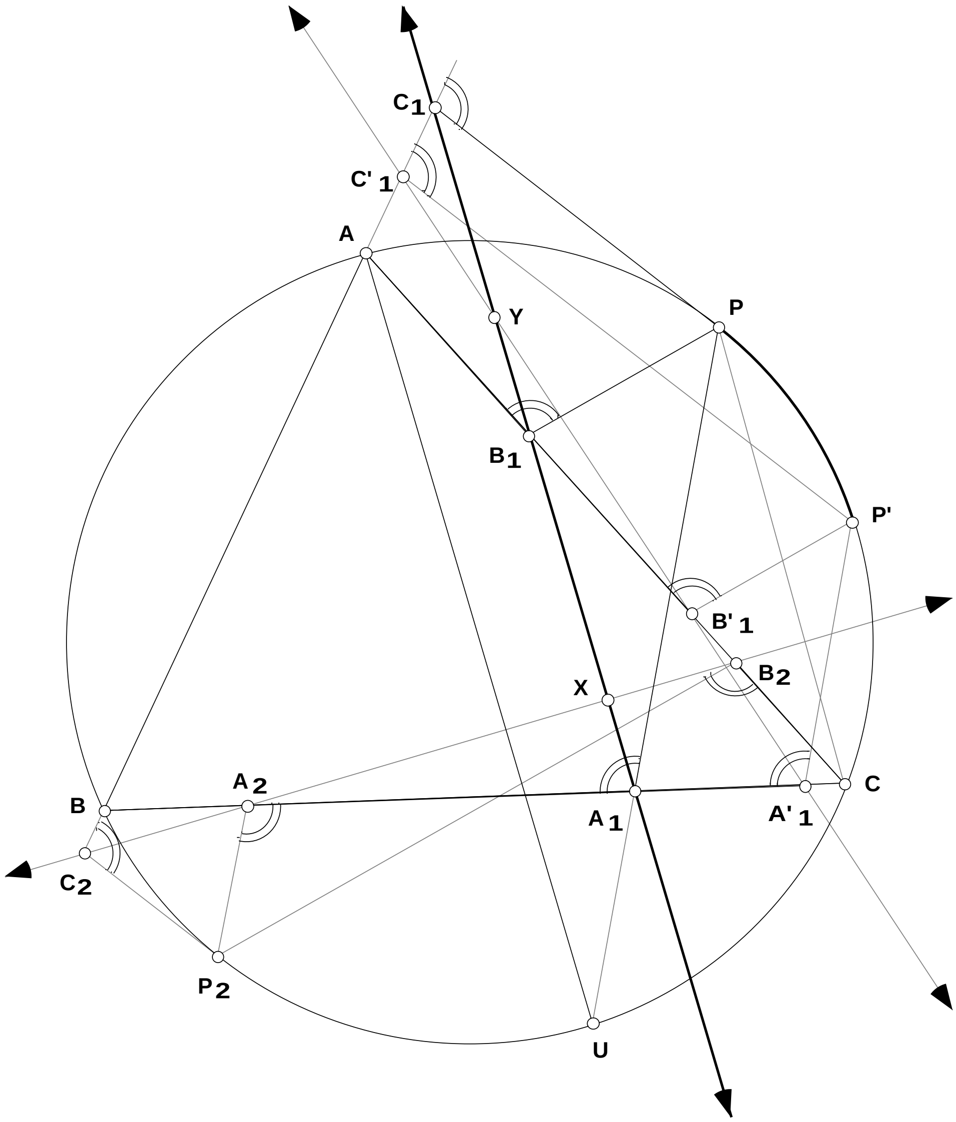

Figure 4

Consider Figure 4, which shows a Miquel line $A_1B_1C_1$ in relation to point *P*. The following results then hold:

(1) If line PA_1 is extended to meet the circumcircle at *U*, then the line *AU* is parallel to the Miquel line $A_1B_1C_1$.

(2) If from another point *P'* on the circumcircle, lines parallel to those from *P*, are drawn to the sides of *ABC* to form another

Miquel line $A'_1B'_1C_1$, then the angle between these two Miquel lines ($\angle C_1YC'_1$) is half the angular measure of the arc *PP'*.

(3) The Miquel lines of diametrically opposed points *P* and P_2 on the circumcircle are perpendicular to each other - provided lines from P_2 to the sides of *ABC* are drawn parallel to those from *P* to the sides of *ABC*. (This result follows directly from (2) above).

(4) Lastly, as shown in Figure 5, from the spiral similarity between the Miquel line and the Simson line, it follows that the Miquel line would also envelope a beautiful deltoid (or Steiner's hypocycloid) as *P* moves around the circumcircle [6].

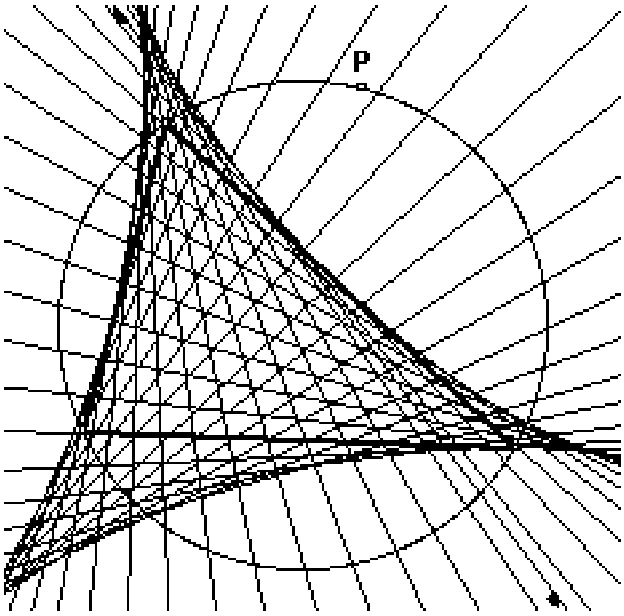

Figure 5

Note

Dynamic Geometry (*Sketchpad 3*) sketches in zipped format (Winzip) of most of the results discussed here can be downloaded from:
http://mzone.mweb.co.za/residents/profmd/miquel.zip

(Note: these sketches may not work properly in *Sketchpad 4* as there is some incompatibility between the two versions. If not in possession of a copy of *Sketchpad 3*, these sketches can be viewed with a free 1 MB demo version of *Sketchpad 3* that can be downloaded from:

http://mzone.mweb.co.za/residents/profmd/gsketchd.zip)

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