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Overlapping Circles[1]

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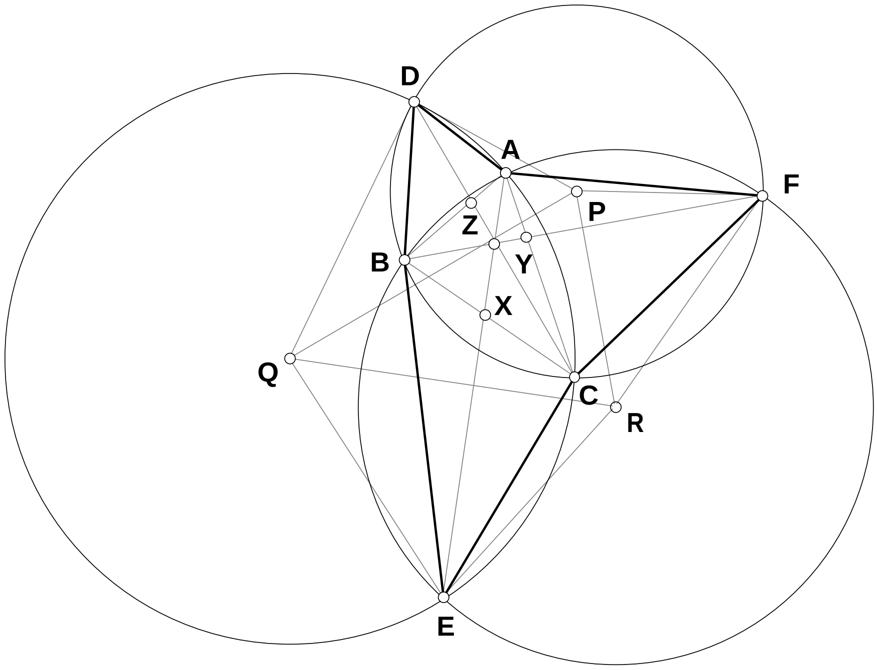
Recently visiting Hilton College, I noticed the following surprising result (without the dotted auxiliary lines) on the cover of a file/book of Sue Southwood:



$$\left(\frac{m\ \overline{BE}}{m\ \overline{EC}}\right)\cdot\left(\frac{m\ \overline{CF}}{m\ \overline{FA}}\right)\cdot\left(\frac{m\ \overline{AD}}{m\ \overline{DB}}\right) = 1.00$$

Intrigued, I immediately tried proving it that evening, eventually coming up with the following proof. Firstly note that the lines *AE*, *BF* and *CD* are concurrent. (These lines are called the power lines of triangle *PQR* formed by the centers of the circles - for a proof consult my book *Some Adventures in Geometry*, pp. 198-199). Let these concurrent lines respectively intersect *BC*, *CA* and *AB* in *X*, *Y* and *Z*. Therefore, according to the theorem of Ceva: $(BX/XC)(CY/YA)(AZ/ZB) = 1$.

Next observe that $\angle BEA = \angle BFA = c$, since they are both on chord *BC* in circle *R*. Similarly, $\angle AEC = \angle ADC = a$ and $\angle CDB = \angle CFB = b$. Applying the sine rule respectively to triangles *BXE* and *CXE*, we obtain:

$\frac{BX}{\sin c} = \frac{BE}{\sin \angle BXE}$ and $\frac{XC}{\sin a} = \frac{EC}{\sin \angle BXE}$. Thus, $\frac{BX}{XC} = \frac{BE \cdot \sin a}{EC \cdot \sin c}$. Similarly, $\frac{CX}{YA} = \frac{CF \cdot \sin c}{FA \cdot \sin b}$ and $\frac{AZ}{ZB} = \frac{AD \sin b}{DB \sin a}$. Thus, $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{BE}{EC} \cdot \frac{CF}{FA} \cdot \frac{AD}{DB} = 1$.

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[1] I recently found out that this theorem is attributed to Hiroshi Haruki by Ross Honsberger (1995). *Episodes in 19th & 20th Century Euclidean Geometry*. Washington: MAA.