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Solutions to Reader Investigations: Feb 2003

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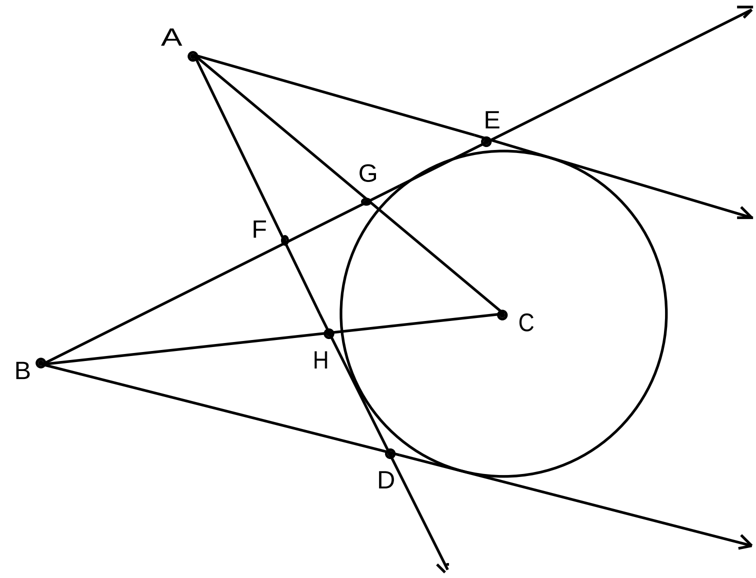
Unfortunately no responses were received to the *Reader Investigations* in the previous issue, but it is hoped that at least a few readers or their learners worked on some of the problems below.



1. Tangents to a circle C are drawn from two arbitrary points A and B outside the circle. Find and prove the relationship between the "center" angle ACB, and the two angles AEB and BDA.

Solution: The relationship is that $\angle ACB = \dfrac{\angle AEB + \angle BDA}{2}$. To prove this is not difficult, but requires a bit of angle calculation. There are several ways of doing it, and the following might not be the shortest possible proof. It is perhaps one of those results were the result is more interesting than the proof is instructive.

In triangle AGE, $180° - \dfrac{\angle DAE}{2} - \angle AEB$ and therefore $\angle BGC = 180° - \dfrac{\angle DAE}{2} - \angle AEB$. Thus in triangle BGC:

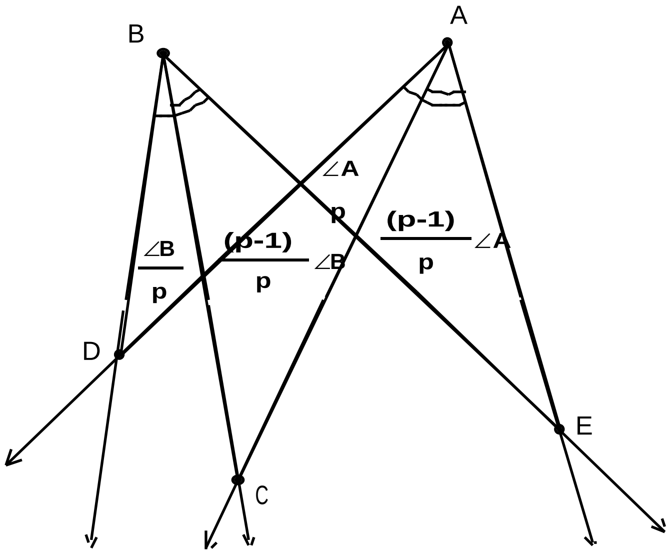
$$\angle ACB = 180° - \left(180° - \frac{\angle DAE}{2} - \angle AEB + \frac{\angle DBE}{2}\right)$$

$$= \frac{\angle DAE}{2} + \angle AEB - \frac{\angle DBE}{2} \quad \ldots(1)$$

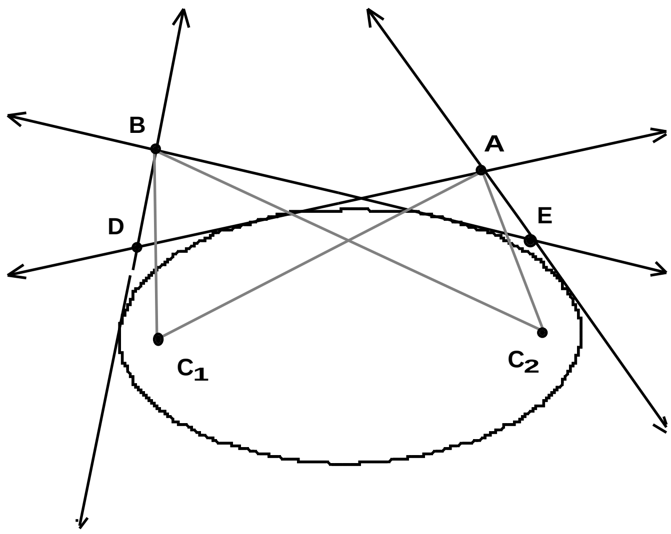
But from the exterior angle of a triangle theorem, we have:

$$\angle DFE = \angle DAE + \angle AEB = \angle DBE + \angle BDA$$
$$\Leftrightarrow \frac{\angle DAE}{2} - \frac{\angle DBE}{2} = \frac{\angle BDA}{2} - \frac{\angle AEB}{2}.$$

Substitution of the latter identity into (1) gives the desired relationship.

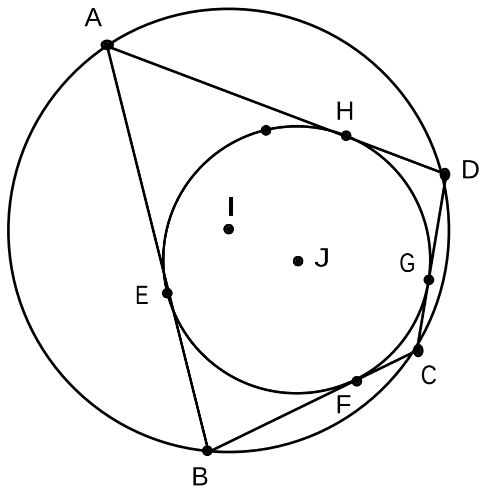


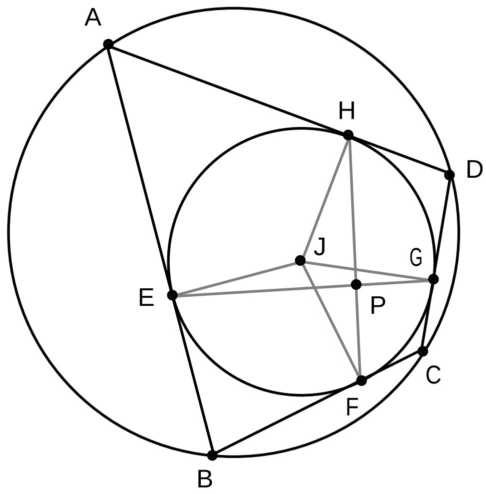
The result and its proof easily generalize to the configuration shown above where the angles are divided as shown. Here the following relationship holds: $\angle ACB = \dfrac{\angle AEB + (p-1)\angle BDA}{p}$.



Perhaps more interestingly, the result can also be generalized to an ellipse as shown above where $\angle AEB + \angle BDA = \angle BC_1A + \angle BC_2A$. However, to prove this result requires some knowledge of the properties of ellipses, but the interested reader may wish to look up a proof of this result which appeared in an issue of the *Mathematical Gazette* of a few years ago. A zipped *Sketchpad* sketch can also be downloaded from:

http://mzone.mweb.co.za/residents/profmd/ellipsetangent.zip





2. Circumscribe a cyclic quadrilateral *ABCD* around a circle. What do you notice about the lines *EG* and *FH* joining the opposite tangential points? Prove your observation.

Solution: The lines *EG* and *FH* are perpendicular to each other. Draw segments as shown in the second sketch above. Note that quadrilaterals *AEJH*, *BFJE*, *CGJF* and *DHJG* are all cyclic. Since *ABCD* is cyclic, angles *BAD* and *BCD* are supplementary, but so are angles *BAD* and *EJH*. Therefore, $\angle EJH = \angle BCD$. In isosceles triangle *EJH*, it follows that $\angle JEH = \angle JHE = 90° - \frac{\angle C}{2} = \frac{\angle A}{2}$ using the useful (but rather sloppy) shorthand of $\angle C$ and $\angle A$ respectively for $\angle BCD$ and $\angle BAD$.

In the same way in isosceles triangles *EJH* and *FJH*, it follows that $\angle JEG = 90° - \frac{\angle D + \angle A}{2}$ and $\angle JHF = 90° - \frac{\angle A + \angle B}{2}$. Therefore, in triangle *EPH*, we have:

$$\begin{aligned}\angle EPH &= 180° - (\angle JEH + \angle JHE + \angle JEG + \angle JHF)\\ &= 180° - \left(\frac{\angle A}{2} + \frac{\angle A}{2} + 90° - \frac{\angle D + \angle A}{2} + 90° - \frac{\angle A + \angle B}{2}\right)\\ &= \frac{\angle D}{2} + \frac{\angle B}{2}\\ &= 90° \ \ldots (\angle B + \angle D = 180°)\end{aligned}$$

The converse is also true, namely, given a quadrilateral circumscribed around a circle with the lines connecting the opposite tangential points perpendicular, then it is also cyclic. The proof, however, is left as an exercise to the reader.

3. (a) Keeping a fixed and varying d, investigate and make a conjecture regarding the graphs of the corresponding family of quadratic functions of the form $y = ax^2 + (a + d)x + (a + 2d)$. (For example, consider the family $y = x^2 + 2x + 3, y = x^2 + 3x + 5, y = x^2 + 4x + 7$, etc.)

(b) Keeping b fixed and varying d, investigate and make a conjecture regarding the graphs of the corresponding family of quadratic functions of the form $y = (b - d)x^2 + bx + (b + d)$.

(c) Keeping c fixed and varying d, investigate and make a conjecture regarding the graphs of the corresponding family of quadratic functions of the form $y = (c - 2d)x^2 + (c - d)x + c$.

Prove your conjectures in (a), (b) and (c) above.

Solution: (a) This is a nice investigation for using the graphing facility of *Sketchpad* to discover that with a fixed and a variable d, this family of parabolas all intersect at (-2; 3a). This can be very easily proved through substitution, for example:

$$\begin{aligned} y &= a(-2)^2 + (a + d)(-2) + (a + 2d) \\ &= 4a - 2a - 2d + a + 2d \\ &= 3a \end{aligned}$$

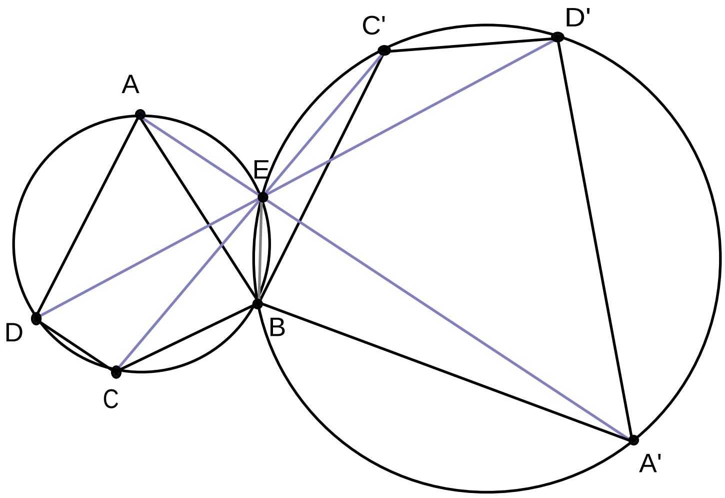
A more instructive approach is as follows. Rewrite the family of parabola in the form $y = a(x^2 + x + 1) + d(x + 2)$. The influence of the parameter d may be neutralized if it is multiplied by 0, that is, if x = -2. In other words, this shows why all parabolas of this family map x = -2 into a real number independent of d.

(b) In the same way can be discovered and proved that this family of parabolas all intersect at the points (1; 3b) and (-1; b).

(c) In the same way can be discovered and proved that this family of parabolas all intersect at the points (0; c) and $(-\frac{1}{2}; \frac{3}{4}c)$.

A set of worksheets that could be used for a classroom investigation of these three families can be downloaded directly from:

http://mzone.mweb.co.za/residents/profmd/work.pdf



4. Two similar cyclic quadrilaterals $ABCD$ and $A'BC'D$ touch each other at the common vertex B. What do you notice about the lines AA', CC' and DD'? Prove your observation.

Solution: Almost without drawing one can visualize that the lines AA', CC' and DD' are concurrent at the other intersection of the two circles. Connect D, C, A, B, C' and D' to E. Then $\angle BED' = \angle BC'D'$ on chord BD' and $\angle BED = \angle BAD$ on chord BD. But $\angle BAD = 180° - \angle BCD = 180° - \angle BC'D'$. Thus angles BED' and BED are supplementary and therefore DED' is a straight line. Similarly, $\angle BEA' = \angle BD'A$ on chord BA'. But $\angle BD'A = \angle BDA$ and therefore in cyclic quadrilateral $AEBD$, $\angle AEB = 180° - \angle BEA'$; thus AEA' is a straight line. Since AEA' is a straight line and $\angle AEC = \angle ABC = \angle A'BC = \angle A'EC'$, it follows that CEC' is also straight.

5. A rectangular prism (box) has integer sides and a surface area of A. Show that A has to be an even integer, and then find all the even values that A *cannot* be.

Solution: Let x, y and z be the dimensions of the box. Then the surface area is $A = 2xy + 2yz + 2xz = 2(xy + yz + xz)$, from which it follows that it must be even. The even numbers that A *cannot be* are called O'Halloran numbers in memory of Peter O'Halloran, an Australian mathematician involved with Mathematical Competitions and Challenges. It turns that there are 16 known O'Halloran numbers (left to the reader to confirm that A cannot be these numbers):

8, 12, 20, 36, 44, 60, 84, 116, 140, 156, 204, 260, 380, 420, 660, 924.

However, it has not yet been established whether there are any more.