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Sketchpad 2001 Competition Solutions

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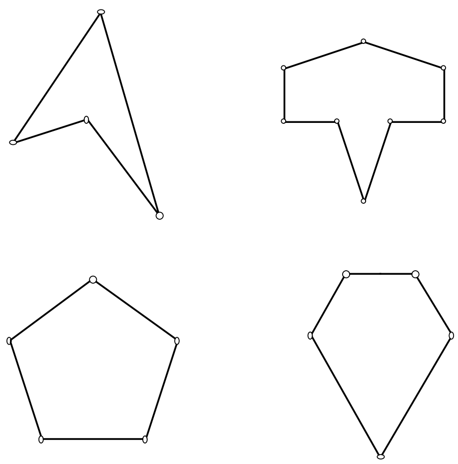
In the July 2001 issue, a copy of the software package *Sketchpad* was offered as a prize for pupils from Grade 9 or lower for the most elegant set of solutions (or partial solutions) for the four problems given below. Unfortunately no satisfactory submissions were received. This is rather disturbing as tessellations is a topic that according to the curriculum, should already be treated in the primary school. Although paper and pencil creation of tessellations is possible, these could be tedious and relatively messy. On the other hand, tessellations can easily be created and explored with the free demo version of *Sketchpad* - which can be downloaded at http://www.keypress.com/sketchpad/sketchdemo.html).

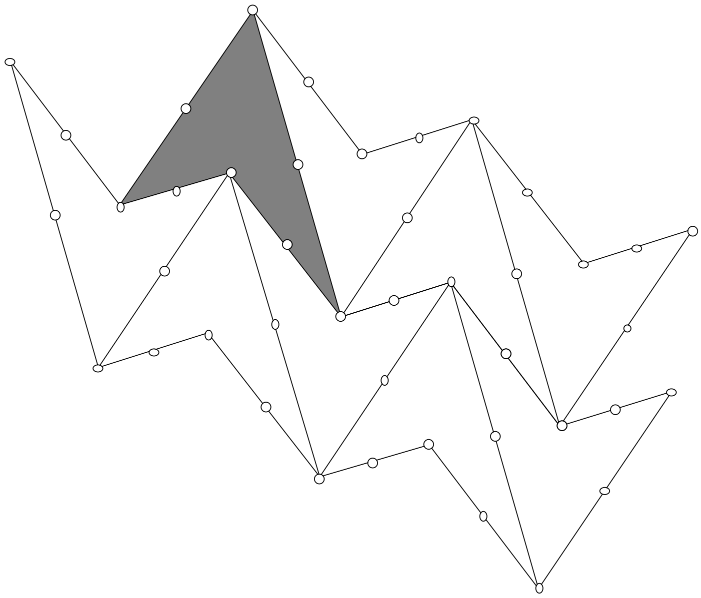
1. (a) Which of the following tiles will tessellate (tile with no overlapping or spaces in between)? Investigate by copying or redrawing the tiles **exactly** to scale. If they do not tile, try to explain why they do not, and if they do, submit neat hand-drawn copies of your tessellation patterns.

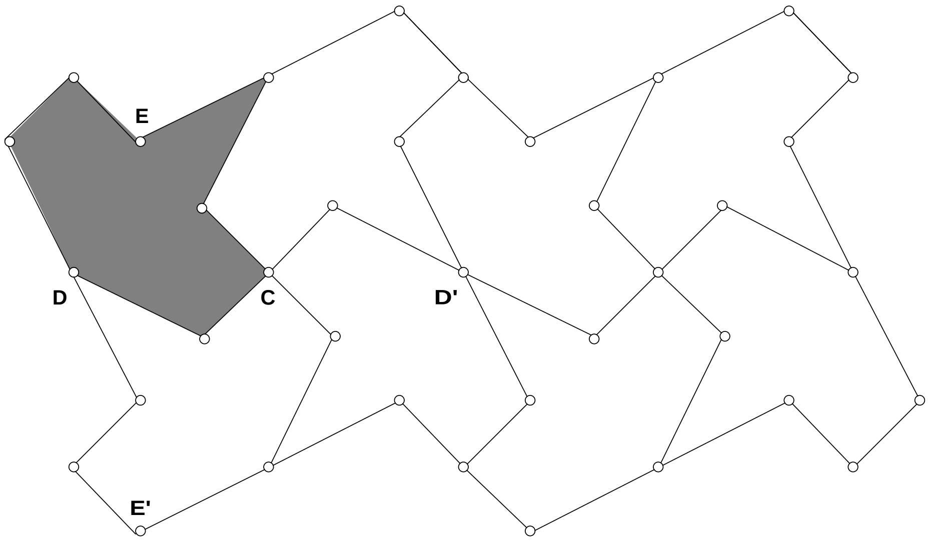


(b) For your tessellation patterns above, also try to identify the rigid motions (translations, reflections or rotations) that could be used systematically to move one tile to another, thus creating the pattern as a whole.

Any quadrilateral tessellates since the sum of its interior angles is equal to 180 degrees, and therefore all four angles can be always placed around a point with no gaps or spaces in between. The tiling pattern with the concave quadrilateral can be created by half-turns (rotation by 180 degrees) around the midpoints of the sides (see below).

The given octagon tessellates as shown below by rotations of 90 degrees around point *C* and then a translation of the group of four tiles by vector *DD*', vector *EE*', etc. This is a very familiar Islamic tiling and appears for example on the Mosque at the University of Durban-Westville. It also appears on the famous Moorish palace Alhambra at Granada in Spain, built between 1248 and 1354.

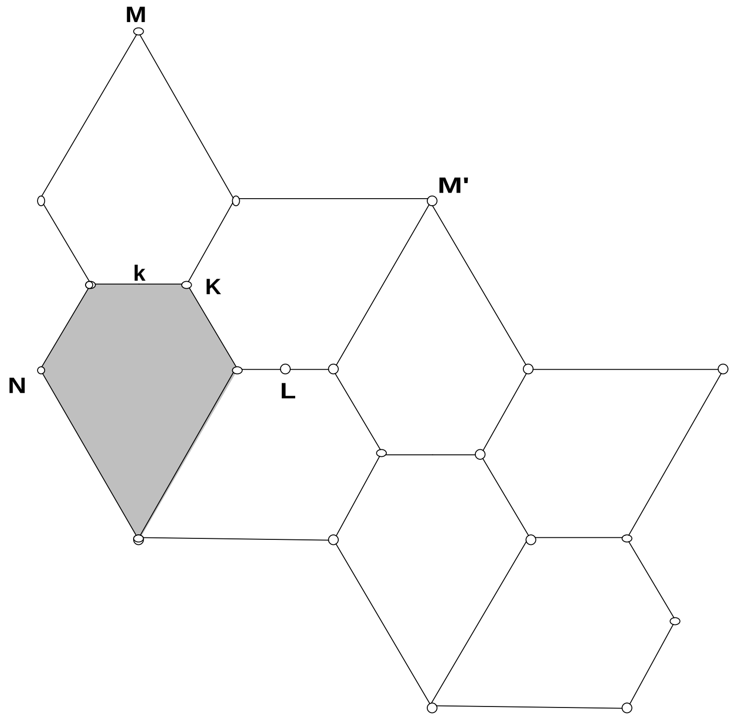




The regular pentagon cannot tessellate as one of its interior angles is equal to 108 degrees and it is therefore impossible to arrange its angles around a point to add up to 360 degrees.

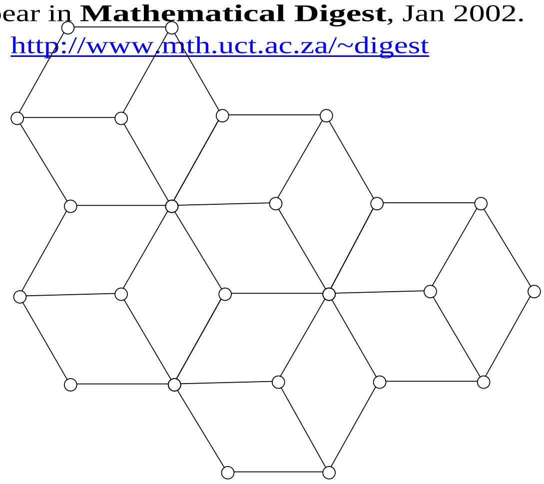
Up until 1918 only five types of tilings with irregular convex pentagons were known. Three more tessellating pentagons were discovered in the late 1960's by R. Kershner, and by 1977 a housewife Marjorie Rice discovered another three. Today 14 different tessellating pentagons are known, but it is still an unresolved problem whether these are the only ones or whether there are more.

The tessellating pentagon below was one those discovered by Kershner. Apart from an axis of symmetry it has 4 angles of 120 degrees (and one of 60 degrees). It can be created by a rotation of the base tile by 120 degrees around point *K* and its reflection in line segment *k*. The 120 degree rotated tile can then be given a half-turn around *L* (the midpoint of one of the sides). Then the group of four tiles can be translated by vectors *MM'*, *MN*, etc. to cover the plane.

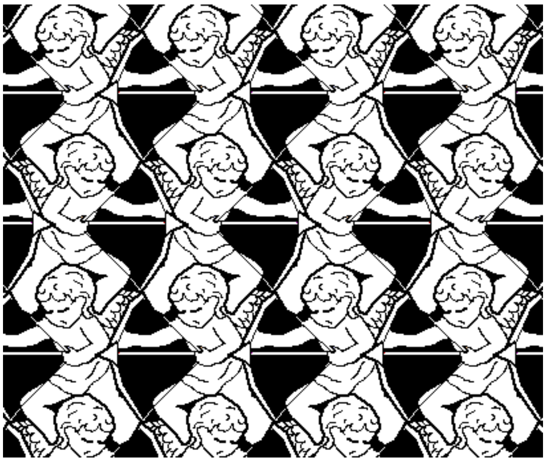


2. Is it possible to create a tessellation pattern with a rhombus tile (a quadrilateral with equal sides) by continually reflecting in its sides? Explain your answer.

Yes, but only if the rhombus's angles are 120 and 60 degrees respectively. See tiling below where the three equal obtuse angles are arranged around a point and therefore have to be 120 degrees each. (Or six equal acute angles are arranged around a point and are therefore 60 degrees each).



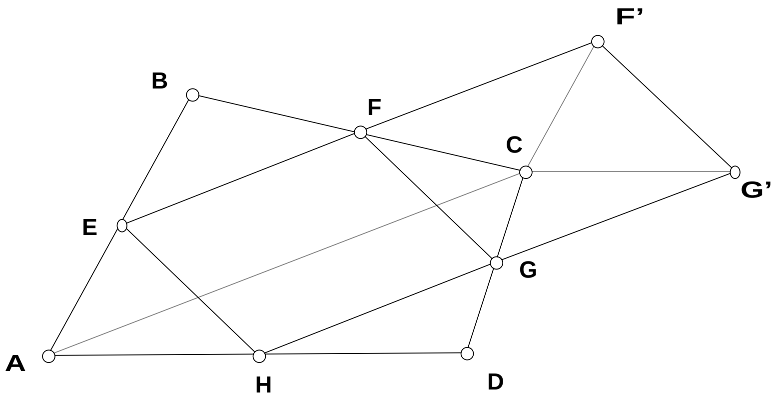
3. For this question learners were required to create their own Escher tessellation-art and try to identify the rigid motions (translations, reflections or rotations) that could be used systematically to move one tile to another, thus creating the pattern as a whole. The computer program *Tesselmania!* (available at R100 + R15 (postage) from the address at the end) was used to create and print out the glide reflection tiling below.



4. (a) What general type of quadrilateral *EFGH* is formed by the midpoints of the sides of an arbitrary quadrilateral *ABCD*?

(b) What is the ratio between the area of *EFGH* and the area of *ABCD*?

(c) Under which conditions will *EFGH* be: (i) a rectangle, (ii) a rhombus, (iii) a square?



(a) *EFGH* is a parallelogram (*EF* //= $\frac{1}{2}$*AC* in triangle *ABC* and *HG* //= $\frac{1}{2}$*AC* in triangle *ADC*; so opposite sides are equal and parallel).

(b) Area *ABCD* = 2 area *EFGH*. (Proof: Translate *EFGH* with vector *EF*. Then triangles *FCF'*, *GCG'* and *CF'G'* are respectively congruent to triangles *FBE*, *GDH* and *AEH*. Triangle *FCG* is common to *ABCD* and *FGG'F'*. Therefore, the four triangles *FBE*, *GDH*, *AEH* and *FCG* lying outside *EFGH* are together equal in area to *EFGH*. This completes the proof.)

(c)(i.) When diagonals are perpendicular.

(ii.) When diagonals are equal.

(iii). When diagonals are both perpendicular and equal.

(Proofs follow directly from 4(a) above).