Published in **Pythagoras**, (2001), No. 53, pp. 14-17, official journal of AMESA.

Game, Math, and Luck!

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Introduction

To many people mathematics and sport are very far apart. Indeed many school children believe that excelling in mathematics and being a sportsperson is on completely opposite poles. The former is simply stereotyped as a "nerd" whereas the latter is labelled a "jock" (or "jill"). Admittedly, there are a few young gifted mathematicians who look down on games and physical training. (See Note 1). However, belying the public misconception, many successful scientists seriously participate in sport as it helps promote and maintain their all-round health, intellectually, psychologically, as well as physically.

On the other hand, sport also provides an inexhaustible source of fascinating and challenging problems in medicine, bio-mechanics, hydro- and aerodynamics, social science, statistics, etc. Today problems like these are increasingly examined, described, and solved by mathematical experts. The purpose of this article is to show one example of how a particular sporting context can be modelled from a mathematical perspective to give us some useful insight. The context that will be examined here is the scoring system of tennis, and what impact a change in that scoring system might have.

A Brief Historical Digression

According to some records, a rudimentary form of the game of tennis already existed in ancient Egypt. In the 13th century, it was played in France where the players threw the ball to each other with the palms of their hands. It was called royal tennis or "game of the palm" ("jeu de paume").

A game of tennis in the mid 1870's is even described in the great Russian novel *Anna Karenina* by Leo Tolstoy, though it bore little resemblance to the modern game. It was played regardless of the ground surface, players wore inconvenient clothes, racquets were loosely strung, and tennis balls were made of pieces of cloth stuffed with horsehair.

Since then tennis has developed into becoming one of the most popular sports in the world. It is played by approximately 120 million people in 193 countries as opposed to only 40 million soccer (football) players world-wide. Its popularity stems from providing immense personal psychological and physical satisfaction, as well as its social aspects. It is different from many other sports in that it requires the endurance of a long-distance runner, the speed of a sprinter, and the swift thinking of a chess player driven in a corner.

The Problem

Recently the South African Tennis Association introduced the so-called "short-deuce" or "no-add" game in the Inter-provincial Veterans Tennis (over 35), as well as in the Junior Tennis Tournaments under its auspices. This means that upon reaching deuce, instead of a player needing to win two consecutive points to win the game, a "sudden death" is played with the player winning the next point, winning the game. Apparently the main motivation is to shorten matches. Whereas this motivation may make sense for (unfit) older players, one must question whether it is in the best interest of young, developing players. Young children actually need to play as much and as long as possible to improve and stimulate their powers of concentration. My interest in the problem comes from being a provincial veteran tennis player myself, and having a son who plays highly competitive junior tennis (see Figure 1).

The first problem with this rule is that it removes a certain dramatic element from tennis: the most absorbing, interesting games in a tennis match are often those in which there are several adds and deuces; first the game is swinging this way, and then it is swinging the other way. A long game with many deuces requires a very high level of concentration and determination to win, and one can only wonder how young children are going to develop these abilities if only exposed to "short-deuces". Another common complaint from particularly older players is that with this new rule "luck" rather than "skill" can determine who wins a game, set or ultimately even a tight match. It is this latter aspect that we will mathematically investigate further.

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Figure 1: My son, Rikus, representing South Africa at the U/12 Nike International Masters, Nov 2001, Bahamas. Photo: Jan Hamman, *Beeld* Newspaper.

A Simple Mathematical Model

The first step in mathematical modelling is to make certain assumptions to simplify the real world context. To keep matters simple we will assume that player A plays against player B, and wins any given point with probability p (and therefore loses any given point with probability $q = 1 - p$).

Let us now examine the combinatorics (all possible outcomes) of a normal tennis game. At the start of the game the score is 0-0. There is only one way for player A to reach scores of 15-0, 30-0 and 40-0 in his/her favour, and that is by respectively winning 1, 2 and 3 points in a row. This gives us the first row in Table 1, and due to symmetry also the first column (for player B). If player A or B wins the next point at 40-0 in their favour, s/he wins the game as indicated. So the probability of A winning "game-0 (love)" is p^4. (See Note 2).

Player B \ Player A	0	15	30	40	game			
0	1	1	1	1	1			
15	1	2	3	4	4			
30	1	3	6	10	10			
40	1	4	10	20	20	20		
game	1	4	10	20	40	40	40	
				20	40	80	80	80
					40	80	160	
						80		

A wins
advantage A
deuce
advantage B
B wins

Table 1: Combinatorics of game

However, there are two ways in which the score can become 15-15. For example, player A could lose the first point and win the next point, or win the first point and win the next point. It is left to the reader to check that there are 3 different ways in which the score can become 30-15 in A's favour, and 4 different ways in which the score can become 40-15 in A's favour. This gives us the second row in Table 1 (and by symmetry the second column for player B). For A to win game-15, he needs to win the next point.

A useful pattern in Table 1 now becomes apparent: the number in each cell is given by the *sum* of the cell directly above it and the cell to the left of it. This allows us to easily complete the rest of the table up to the first deuce (which can be reached in 20 different ways!) The probability of A winning "game-30" is therefore $10p^4q^2$. So the probability of A winning "game-15" is $4p^4q$. The **total** probability of A winning the game is $p^4 + 4p^4q + 10p^4q^2$, plus whatever happens after deuce. (See Note 3).

Deuce complicates things a bit. To reach the first deuce each player needs to win 3 points each. After deuce, a player needs to win another two points in succession to win the game. For example, the probability of A winning after the first deuce is $20p^5q^3$. However, if A wins the first point and loses the next (or vice versa) we get the second deuce where each player has won and lost 4 points each (and which can be reached in 40 different ways). Therefore, the probability of A winning after the second deuce is $40p^6q^4$.

Now note how the numbers representing wins for A (from "game-30") run down a diagonal: 10, 20, 40, 80, 160, ... doubling all the time. In other words, the total probability of A winning the game is:

$$p^4 + 4p^4q + 10p^4q^2 + 20p^5q^3 + 40p^6q^4 + 80p^7q^5 + ...$$

$$\Leftrightarrow p^4 + 4p^4q + 10p^4q^2(1 + 2pq + 4p^2q^2 + 8p^3q^3 + ...)$$

But the infinite series in the brackets is simply a *geometric progression* with $r = 2pq$. Using the formula for the sum to infinity of a geometric progression (see Note 4), we obtain the exact total probability of A winning the game as:

$$p^4 + 4p^4q + \frac{10p^4q^2}{1-2pq}.$$

Suppose player A wins every two points out of three, i.e. $p = \frac{2}{3}$. Does this mean that A will win about 2 out of every 3 games played?

Perhaps surprisingly, this is not the case at all. According to the formula, A's total probability of winning a game would then be $\frac{208}{243} = 0.856$ which is about $\frac{6}{7}$. In other words, A will win about 6 games out of every 7 games played! The rules of tennis are therefore specifically designed to **amplify** differences between players, and to favour the "better" player. The first two columns in Table 2 gives the probability of winning a game for several different values of *p*. From these values one can see how the "better" player is consistently favoured. Note that as the "quality" of the player increases (decreases) above 0.55 (below 0.45), there is a very sharp increase in the probability that s/he would win (lose) the game. This effectively eliminates the possibility that a much weaker player can win a game on sheer "luck". For example, a player who wins only 3 out of every 10 points (p = 0.3) has a chance of winning a game by sheer "luck" of only 0.099 or about one in ten games.

p	Game	sd-Game
0	0	0
0.1	0.0014	0.0027
0.2	0.0218	0.0333
0.3	0.0992	0.1260
0.4	0.2643	0.2898
0.45	0.3769	0.3917
0.5	0.5	0.5
0.55	0.6231	0.6083
0.6	0.7357	0.7102
0.7	0.9008	0.8740
0.8	0.97822	0.9667
0.9	0.9986	0.9973
1.0	1.0	1.0

Table 2

The "Short-Deuce" Game

What are the combinatorics of a "short-deuce" game, and the probability of winning it? Is it really true that it introduces a greater element of "luck"?

In contrast to a normal game, the game is won by the player who wins the first point after deuce. Looking back at Table 1, it therefore follows that the probability of winning a "short-deuce" game simply becomes:

$$p^4 + 4p^4q + 10p^4q^2 + 20p^4q^3.$$

Suppose player A again wins every two points out of three, i.e. $p = \frac{2}{3}$. Then according to this formula, A's total probability of winning the game is equal to $\frac{1808}{2187}$ = 0.827 which is about $\frac{5.8}{7}$. So in a short-deuce game, A would win about 5.8 games out of every 7 played, whereas in a normal game it is about 6 games. Thus, although the probability that A would win a game has decreased (implying that his weaker opponent now has a better chance of winning the game), the decrease is perhaps not as much as one might have expected.

A graph showing a comparison between the probabilities of winning a normal game and a short-deuce game is given in Figure 2. Interestingly both graphs have the shape of a typical titration curve in chemistry, i.e. are very flat at each end, but rise steeply in the middle. This implies that with a probability of more than 0.7 (or less than 0.3) of winning each point, one's chance of winning (or losing) a game, rapidly approaches 1 (or 0).

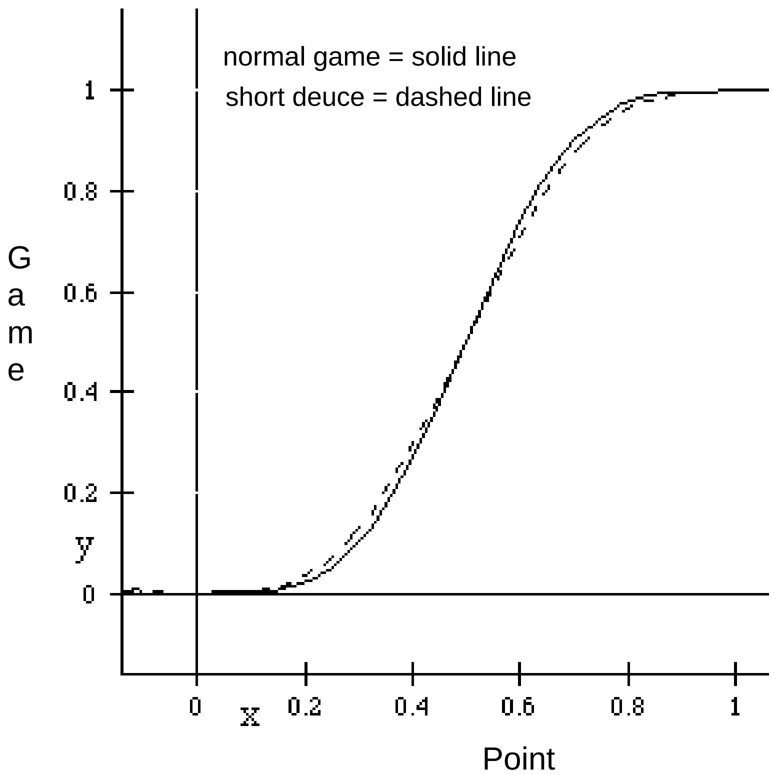

Figure 2

The third column in Table 2 gives the probability of winning a short-deuce (s-d) game for several different values of p. From these values, as well as the graphs, one can see how the "better" player is still consistently favoured, although his/her winning probability is somewhat lower. Correspondingly there is also a slight increase in the possibility that a much weaker player can win a game on sheer "luck". For example, a player who wins only 3 out of every 10 points (p = 0.3) has a slightly increased chance of 0.126 of winning a short-deuce game by sheer "luck" or about 1.3 games out of every ten games.

From Figure 2 and Table 2, one can clearly see that the short-deuce game is not likely to have much effect when one player is substantially better than the other.

However, the effect of increased "luck" is somewhat more of a concern during a more evenly contested match. For example, consider two players with respectively p = 0.45 and p = 0.55. In a normal game the respective game winning probabilities are 0.3769 and 0.6231. However, in a short-deuce game the respective game winning probabilities are 0.3917 and 0.6083. This narrowing of the probability "gap" between two players is certainly a bit more problematic when players are very evenly matched than when one player is substantially better than the other.

Concluding Remarks

The above analysis has shown that the short-deuce game does indeed increase the element of "luck" in winning a game, but it is perhaps less than expected. This effect is only likely to be felt in closely contested matches and hardly at all in one-sided affairs.

Although this mathematical model gives some interesting insight, it (like any other mathematical model) only approximates reality. In particular, the assumption that the probability of winning a point is always the same is not realistic. For example, when a player is serving s/he may stand a better chance of winning a point than when receiving. (This may not necessarily be true for very young players who have not yet developed big serves). Some players also have the ability to raise the level of their game on important points while others may choke. Some players play better on certain surfaces, and others worse under variable conditions (e.g. wind, sun, heat, crowd, etc.) Indeed, to mathematically model all the diverse factors that go into tennis, is virtually impossible.

The above combinatorial analysis can be extended to determine the probability of winning a "best of 3 tie-breaker sets" match, but the mathematics gets a little too cumbersome for the intention of this article (compare Sadovskii et al, 1993). Interestingly, the differences between players are even far more strongly emphasised when sets and matches are brought into account. For example, if a player only has $\frac{1}{3}$ chance of winning a point, his/her chance of winning a 3 set match is only 0.000000027, or about one in thirty-seven million! Even so, in a closely contested match between more or less even opponents, best of five sets rather than just three sets, is necessary to give favourable odds to the "better" player (compare Sadovskii et al, 1993).

Notes

1. In a study by Leder & Taylor (2000) it was found that 355 of 551 past medal winners (64%) in the Australian Mathematics Competition participated in some sport or another. This refutes the historical stereotype of the single mindedness of those capable of outstanding achievement in mathematics. Similarly, personal experience with medal winners of the South African Mathematics Olympiad, also testifies to a high percentage of active sport involvement among them (besides many other interests).
2. The probability of n independent events happening *in turn* is given by the *product* of the individual probabilities.
3. The probability of a set of independent events is given by the *sum* of the individual probabilities.
4. $a + ar + ar^2 + ar^3 + ... = \frac{a}{1-r}$, provided $|r| < 1$.

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