Transformations: A golden thread in school mathematics

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Introduction

The latest syllabus proposals for implementation in 1992 describe, among other things, the following new content:

- tessellations in Std 5
- scale drawings and similar figures in Std 6
- isometric and similar transformations of plane figures as an optional topic in Std 7.

The purpose of this article is to place the above topics in a broader perspective and to illustrate the concepts of transformation geometry as a possible golden thread, not only throughout geometry, but also in relation to algebra and trigonometry.

Tessellations

Tessellations are a most welcome addition to primary-school mathematics, and can mainly be justified from the following viewpoints:

1 Tessellations provide an intuitive visual foundation for a variety of geometric content which is later treated formally in a logico-deductive context
2 Tessellations have a great aesthetic attraction due to the intriguing and artistically pleasing patterns that can be created with them (e g Escher's work)
3 Tessellations comprise an interesting mathematical topic in its own right which can provide an excellent opportunity for self-exploration and — analysis (Olivier, 1990).

The first viewpoint is strongly supported by the Van Hiele theory (Van Hiele, 1973) which sees the intuitive visual exploration of space as a prerequisite for the learning of geometric terminology, the development of deductive thinking and the understanding of the nature of definitions and formal axiomatics. For instance, tessellations provide a meaningful context for the informal acquaintance of several important geometric results and concepts.

For example, formal properties of parallel lines such as equal corresponding and alternate angles can be informally introduced in tessellations as '*ladders*' and '*saws*' (Fig 1). Similarly Fig 2 informally illustrates the exterior angle theorem, as well that the sum of the angles of a triangle is 180° and the theorem that DE = BC/2 and DE//BC if D and E are the midpoints of sides AB and AC of ΔABC. Furthermore, besides the implicitly embedded concept of congruency, tessellations can also provide an introduction to the concept of similarity and similar figures as shown in Fig 3. Pupils may also deduce several properties of plane figures from tessellations, for example, the equality of opposite sides and angles of a parallelogram as shown in Fig 3.

A further advantage of tessellations is that they can help develop pupils' visualization ability, i e that they are able to recognize geometric configurations (eg equal corresponding angles) in a variety of orientations (not just 'conventional' ones) — an ability presently sadly lacking in many senior pupils. It is therefore important that tessellations are not seen only as an artform, but as an essential prerequisite for

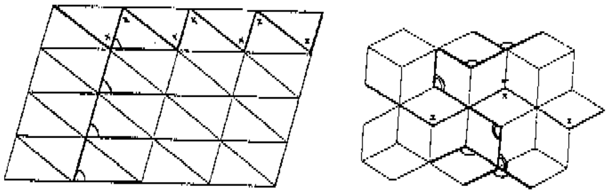

Fig 1

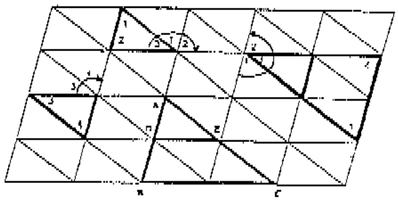

Fig 2

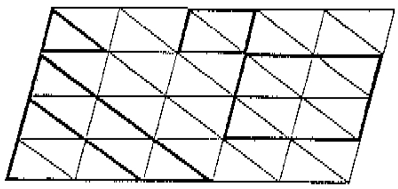

Fig 3

the enrichment of pupils' geometric experiential background in preparation for formal geometry. However, it is probably true that those with a greater inclination towards art than towards mathematics might find tessellations appealing mainly from an artistic point of view (see for example the worksheets 'Kunstige Tessellasies' in *Pythagoras* No 25, April 1991, pp 47–48).

On the other hand, tessellations can provide an excellent opportunity for independent mathematical exploration. In fact, it would be sad if this part of the new syllabus would be treated in the traditional 'talk-and-chalk' approach, in other words, direct one-way teaching from the teacher to the pupils without actively and constructively involving them. Basically two fundamental types of questions can be investigated in relation to tessellations: (1) 'Which types of tiles will cover the plane with replicas of themselves without overlapping or leaving gaps?' (2) 'What movements or motions (transformations) could or must be applied to the basic tile(s) to create particular tessellation patterns?'

For example, children may be asked to investigate which regular polygons tessellate, thus finding that only three possibilities exist, namely equilateral triangles, squares and regular hexagons. Similarly they could be asked to investigate which non-regular polygons tessellate, finding the perhaps surprising result that all triangles and quadrilaterals tessellate, but only some polygons with five sides or more. For example, regular pentagons do not tessellate, but any pentagon with one pair of parallel sides will. Regular hexagons tessellate, but many other hexagons don't. Convex polygons with seven or more sides cannot tessel-

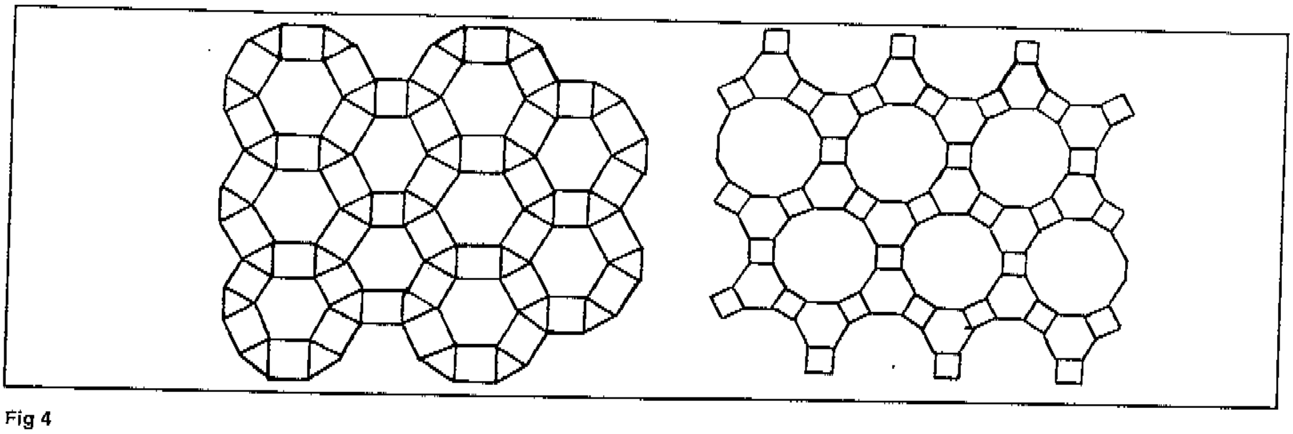

Fig 4

late. This investigation could also be extended to the semi-regular tessellations (i e in which *combinations* of regular polygons are tessellated), two examples of which are shown in Fig 4.

Regarding the second question above, children could be asked to identify which combination(s) of reflections, rotations and translations would produce a particular tessellation pattern. For example, the triangular tessellation patterns given in Figs 1–3, can be produced by starting with the basic triangular tile, and giving it a half-turn around the midpoints of each side, and repeating the process with the newly formed tiles. Or alternatively, we can consider it as a triangular tile which has been given a half-turn around the midpoints of its sides to form a parallelogram, which is then translated in directions parallel to its sides to cover the plane. The first figure in Fig 5 gives a triangular tessellation pattern which could be produced by half-turns of the triangular tile around the midpoints of the sides as indicated to form the first row, and the consequent reflections thereof in a vertical direction as shown. The second tessellation pattern in Fig 5 with kites, could be produced by half-turns around the midpoints of the longer sides in a horizontal direction as shown, and the consequent translations thereof in a vertical direction. Or alternatively (and more economically), the kite tessellation pattern could be produced by only using half-turns around the midpoints of all the sides.

In fact, tessellations still provide an active research area for mathematicians, as testified by the comprehensive study by Grünbaum and Shepherd as recently as 1986 entitled Tilings and Patterns, in which they not only provide a systematically formalized summary of proven results, but also indicate several as yet unanswered questions.

Transformations

It seems that the main aims of the inclusion of transformations in the junior-secondary phase are:

(1) to use transformations in informal geometry to deduce properties of figures (eg folding around lines of symmetry, rotating, etc)
(2) to present motion geometry (transformations) as an alternative mathematical approach to that of traditional Euclidean geometry.

The purpose of this article is, however, to illustrate how the introduction of transformations in the junior-secondary phase can also provide us with a useful aid to treating certain traditional topics in the higher standards in alternative (and sometimes easier and more elegant) ways. A short summary of transformation geometry from an advanced viewpoint is now given:

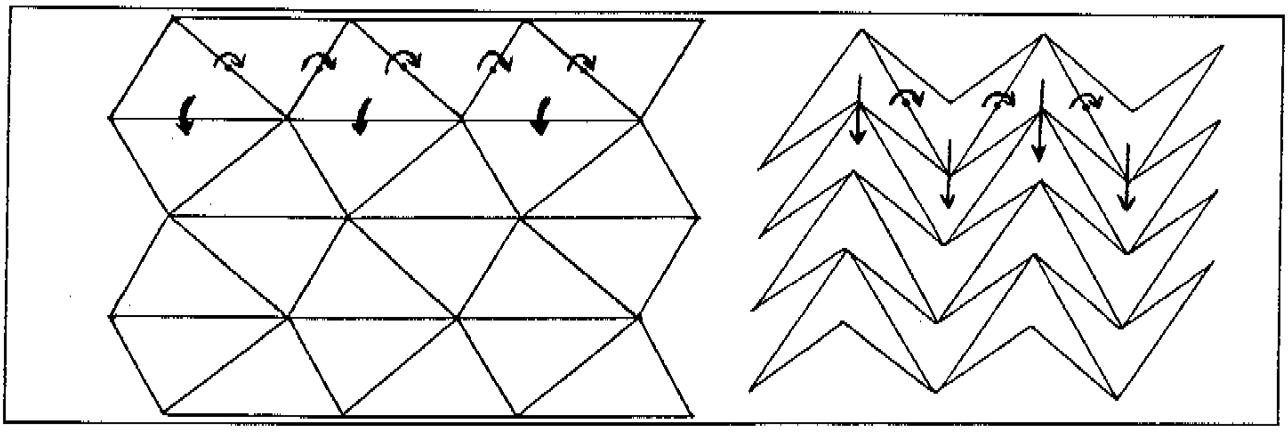

Fig 5

It may even be possible that some of our own pupils could make original contributions to the study of tessellations similar to those of the housewife Marjorie Rice who in the seventies discovered four new convex pentagons that tessellate, although mathematicians had thought at that stage that the list of tessellating pentagons was complete (Schattschneider, 1981).

- isometric transformations (rigid motions)-transformations of plane figures so that distances and angles are preserved (congruency). An example of the three basic isometries, namely reflections, translations and rotations are shown in Fig 6. (A fourth isometry, namely a glide reflection can be distinguished if we want to define the isometries as a group)
- similar transformations — transformations of plane figures where distance is not necessarily preserved, but only angles (and ratios of corresponding sides). An example of

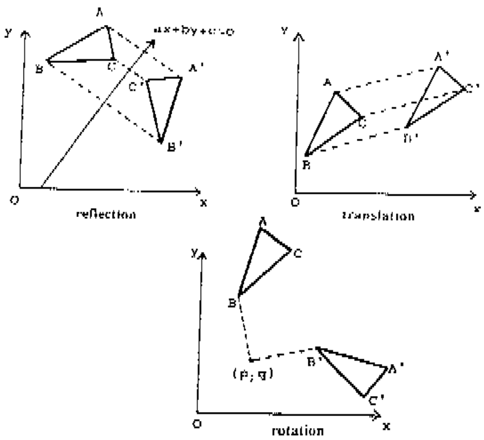

Fig 6

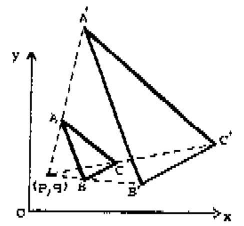

Fig 7

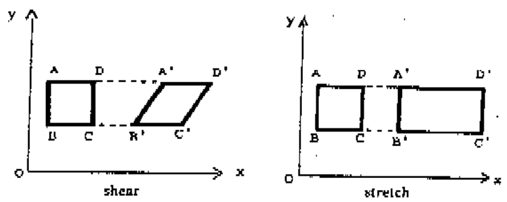

Fig 8

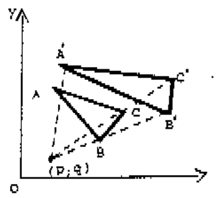

Fig 9

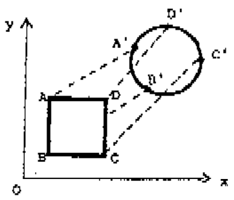

Fig 10

an enlargement is shown in Fig 7

- affine transformations — transformations of plane figures with the only restriction that the parallelism of corresponding lines is preserved. A shear preserves area as shown by the first figure in Fig 8, while a stretch as shown by the second figure, does not
- projective transformations — transformations of plane figures where the collinearity of points and the concurrency of lines are preserved. An example is shown in Fig 9
- topological transformations — transformations of plane figures where the closure (or non-closure), orientability and relative position of corresponding points are preserved (see Fig 10).

Note that the above transformations are hierarchically arranged from the special to the more general cases, in other words, the isometries are a subset of the similarities, which in turn are a subset of the affinities, etc. This approach to geometry is known as Klein's *Erlangen*-approach and refers to a lecture by him in 1872 where he described geometry as the study of those geometric properties which remained invariant (unchanged) under the various groups of transformations.

It's important to note that transformations are also functions, since transformations of the plane are defined as one-to-one mappings of points in the plane onto points in the plane which also implies the existence of inverse transformations. This view of transformations thus encourages a spontaneous and natural blending of algebraic concepts with geometric ones.

It's a good exercise to try and derive the corresponding analytical equations for the above transformations. Although some are relatively difficult in general form, many are extremely easy and should seriously be considered at school level. Consequently some examples:

Reflection in y-axis
$x^1 = -x$
$y^1 = y$

Reflection in x-axis
$x^1 = x$
$y^1 = -y$

Reflection in y = x
$x^1 = y$
$y^1 = x$

Reflection in y = c
$x^1 = x$
$y^1 = 2c - y$

Reflection in x = c
$x^1 = 2c - x$
$y^1 = y$

Translation
$x^1 = x + a$
$y^1 = y + b$

Half-turn around origin
$x^1 = -x$
$y^1 = -y$

Rotation θ around origin (anti-clockwise)
$x^1 = x\cos\theta - y\sin\theta$
$y^1 = x\sin\theta + y\cos\theta$

Enlargement from origin
$x^1 = kx$
$y^1 = kx$

Enlargement from (p; q)
$x^1 = k(x - p) + p$
$y^1 = k(y - q) + q$

Shear in x-direction
$x^1 = x + ky$
$y^1 = y$

Stretch in x-direction
$x^1 = kx$
$y^1 = y$

Shear in y-direction
$x^1 = x$
$y^1 = y + kx$

Stretch in y-direction
$x^1 = x$
$y^1 = ky$

Note that an enlargement with a scale factor k so that
$0 \leq |k| < 1$
gives a reduction. (What happens when $k < 0$?) In the derivation of the analytic equations for a rotation around the origin, formulae for sin(A+B) and cos(A+B) are required, and therefore by first presenting this problem (and solving it until these formulae are required for further simplification), one can provide a *meaningful motivating* context for the consequent derivation of such formulae.

A topic which is perfectly suitable at school level is the application of the isometries to the study of border or strip patterns (in other words, covering a one-dimensional border with a repeating pattern of congruent figures). Three different examples of border patterns constructed with the same basic figure are shown in Fig 11. The mathematical problem is to invent a classification system for such border patterns. Although one may intuitively feel that there is an extremely

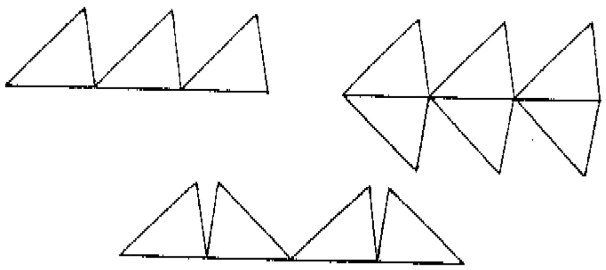

Fig 11

large number of possibilities, only seven basic possibilities exist: a fact long since known by artists, potters and embroiders. Two pottery examples, respectively from San Ildefonso Pueblo (New Mexico) and India are shown in Fig 12(a) (Crowe, 1986: 20–21). Two local examples of the isometries in rural paintings, beadwork and baskets from Kitto (1990) are shown in Fig 12(b) and can provide an excellent opportunity to *multi-culturalize* our mathematics education.

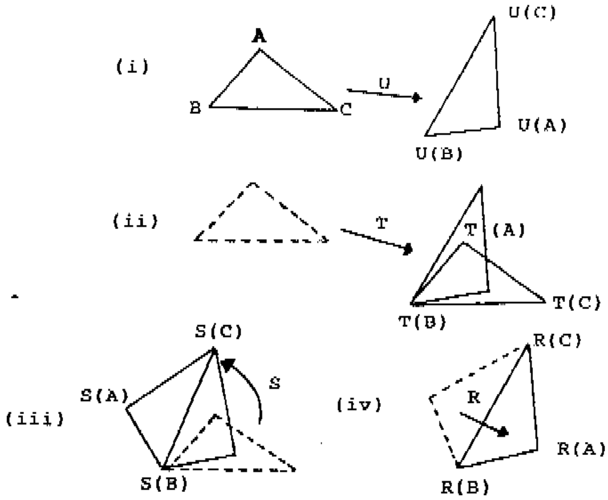

Fig 13

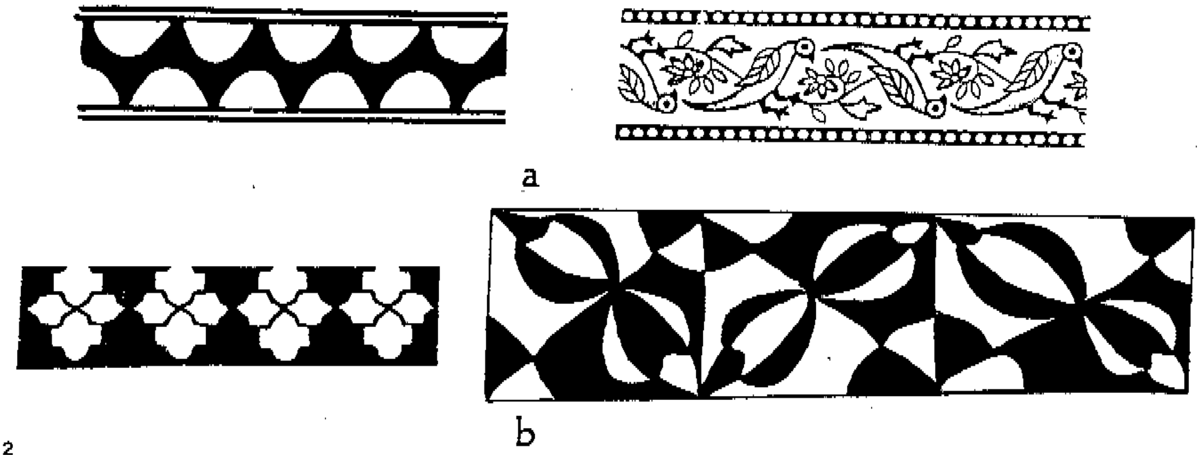

Fig 12

Since pupils can easily explore the classification problem by trial and error in a practical manner, they usually find it interesting and stimulating. However, the more ambitious problem of finding (by trial and error) the total number of possible ways of covering a plane with tessellating figures using only the isometries, probably is best left for the more gifted pupils.

The informal study of some of the properties of the isometrics can also be very interesting. For example, pupils may be led to experimentally discover that any (rigid) motion of a figure in the plane can be obtained by at most three consecutive applications of the basic isometries, namely a translation, a rotation and a reflection. Consider for example the diagrammatic representation in Fig 13:

(i) Suppose we began with ΔABC and applied any rigid motion in the plane U to it
(ii) Then by means of a translation T we can map T(B) onto U(B)
(iii) Using a rotation S around U(B) we can map S(C) onto U(C)
(iv) Lastly, using a reflection R in the line U(B)U(C) we can map R(A) onto U(A).

Other meaningful activities which could be utilized for an introduction to the construction of perpendicular bisectors is to ask pupils to find a way of finding the axes of reflection or centres of rotation when the initial and end positions of figures under these isometries are given. For example, find the axis of reflection for the first transformation in Fig 14 and the centre of rotation for the second one.

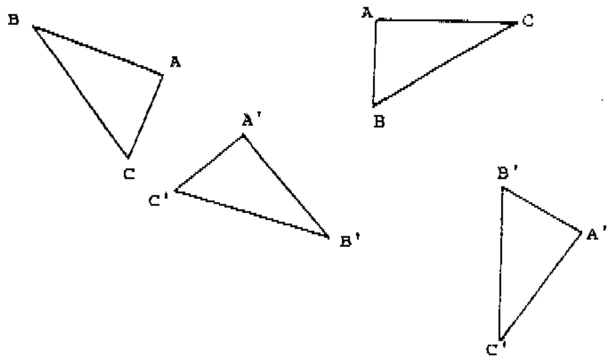

Fig 14

Using transformations to prove geometric results

It would be a great pity if the introduction of transformations in the junior-secondary phase as an optional topic, does not also lead to a radical re-evaluation of the geometry content and goals of the senior-secondary syllabus. For example, the meaningfulness and continued presence of long, unelegant and non-intuitive traditional proofs should be questioned in cases where simpler and more elegant transformation proofs exist. Consider for example Fig 15 and contrast the following proof using *symmetry* with the traditional one based on *congruency*.

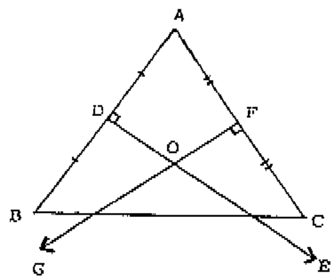

Fig 15

To prove:

The perpendicular bisectors of any ΔABC are concurrent.

Construction:

Draw DE and FG the respective perpendicular bisectors of AB and DC to intersect in O as shown.

Proof:

We now want to prove that O lies on the perpendicular bisector of BC. DE is, however, an axis of symmetry of AB, therefore there exists a reflection around DE so that A and B coincide. According to the definition of a reflection DE is therefore the locus of all the points equidistant from A and B. FG is similarly the locus of all the points equidistant from A and C, and therefore the point of intersection of DE and FG must be equidistant from B and C. In other words, O lies on the axis of symmetry of BC, or rather its perpendicular bisector. QED

A similar proof can also be given for the concurrency of the angle bisectors of a triangle. Besides being shorter and less clumsy than the traditional proofs, these two proofs seem to better satisfy pupils' needs for explanation and verification (De Villiers, 1988; 1990).

Even though alternative proofs with transformations are not always easier or more explanatory, they could nevertheless broaden pupils' perspectives and provide them with greater insight. Several of the circle theorems provide a useful context for the application of transformation geometry. For example, if we start with the configuration shown in Fig 16, it should be intuitively clear that any pair of parallel chords in a circle are symmetrical with respect to the perpendicular bisector of those chords (the diameter), and that the arcs cut off from the circumference of the circle by these chords, are equal.

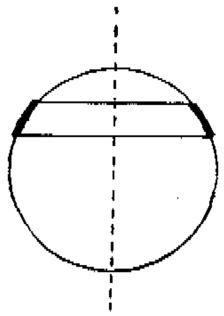

Fig 16

Suppose we now considered two lines PA and PB which form $A\hat{P}B$ on the circumference as shown in Fig 17(a), and we now translate the whole configuration so that P shifts to Q on the circumference. Call the points where PA and PB respectively cut the circle C and D. Then according to the

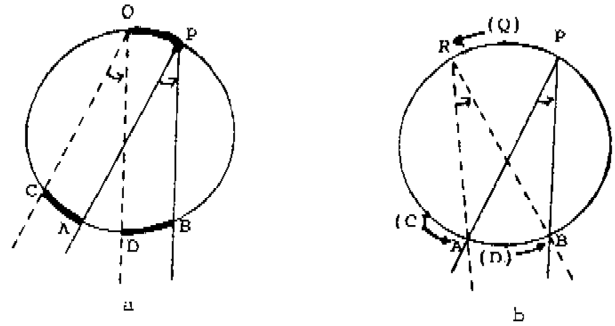

Fig 17

properties of a translation, QC||PA and QD||PB and therefore according to the property in Fig 16, arcs DB, QP and CA are all equal. A rotation around the centre of the circle does not affect the circle, and if we choose the angle of rotation for the new configuration so that C→A, then D→B and Q→R (Fig 17(b)). It therefore follows that $A\hat{R}B = A\hat{P}B$ on the same chord AB.

A very attractive aspect of the above approach is that the same method can be used for other related circle theorems by appropriately choosing a position for Q. Fig 18 for example shows this approach applied to the alternate segment theorem. The theorem that the exterior angle of a cyclic quadrilateral is equal to the opposite interior angle, can also be proved in a similar fashion (Bailey et al, 1974: 102).

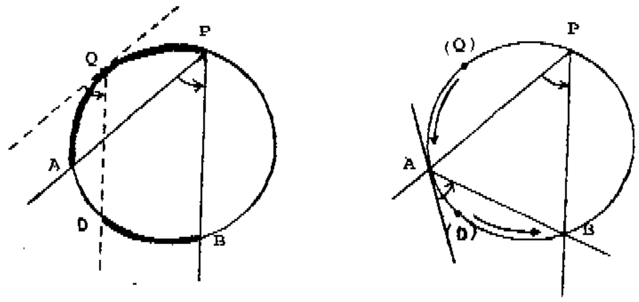

Fig 18

Wynne Wilson (1977: 63) gives another excellent example of the use of transformations in proving the well-known theorem of Ptyhagoras as shown in Fig 19 (in the classic 'peacock' or 'windmill'-configuration of Euclid's original proof). The proof comprises the division of the square on the hypotenuse into two rectangles as shown. Then it is shown that the area of the smaller rectangle is equal to the area of the square on the shorter rectangular side by simply using a shear, a rotation and another shear. The *shear*-transformation is of course generally very useful in any proofs involving the areas of figures, as well as the deduction of the area-formulae for various geometric figures. It is therefore unfortunate that simple examples of the shear-transformation have not been included in the new syllabus for 1992/3.

Another informative proof technique which could also be sonsidered is the utilization of the affine equivalence between certain geometric configurations in order to prove a general statement by the consideration of a particular case (eg consult Shigalis, 1989). Just as two figures are congruent or similar if an isometry or a similarity exists that maps the one onto other, two figures are affinely equivalent if an affine transformation exists that maps the one onto the other. Despite the preservation of parallelism of lines, the affinities also preserve the incidence of points and lines,

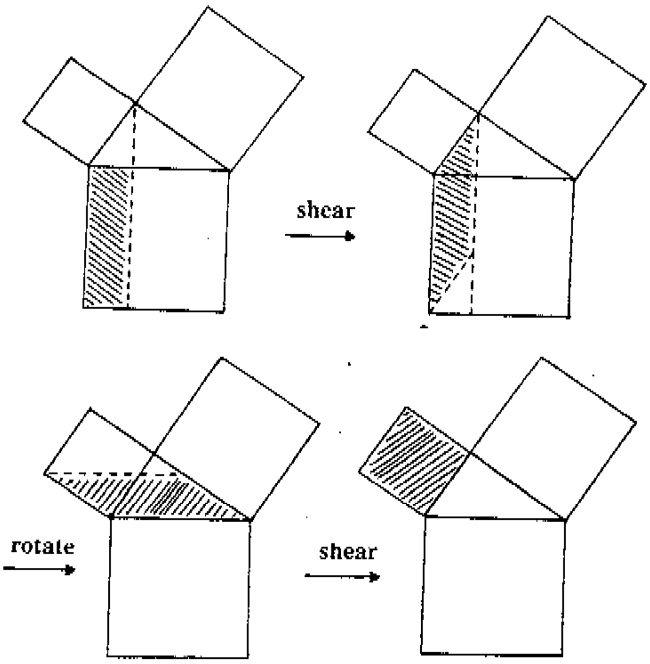

Fig 19

collinearity of points, concurrency of lines and most important the ratio in which a point divides a line segment. For example, the first two figures in Fig 20 are affinely equivalent and involve the theorem which states that the segment joining the midpoints of two sides of a triangle is parallel to the third side and half as long. Since this theorem involves affine relationships and these two figures are affinely equivalent, the general case can be proved simply by showing it is true in the special case on the right (eg by using analytical geometry). Similarly, in the second set of affinely equivalent figures the special case on the right can be used to prove the theorem that the medians of a triangle are concurrent in a point two-thirds of the way from each vertex to the midpoint of the opposite side.

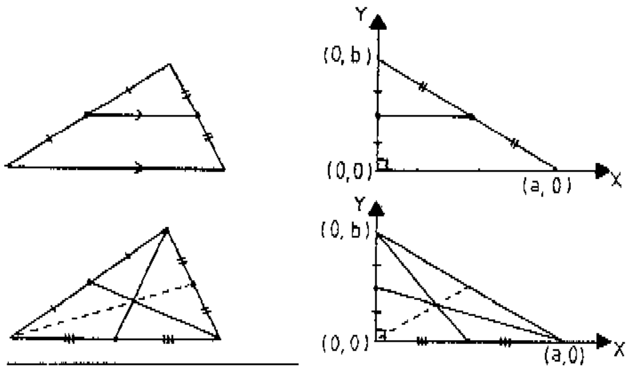

Fig 20

Using transformations in the formal definitions of geometric figures

The utilization of transformation concepts in the formulation of the definitions of some geometric figures, often results in simplifying the deductive derivations of the other properties from the corresponding definitions. For example, consider the following two definitions for a parallelogram and a kite respectively (see Fig 21):

- A parallelogram is any quadrilateral with a half-turn symmetry (or is point symmetric) at the intersection of its diagonals (or midpoint of one of its diagonals)

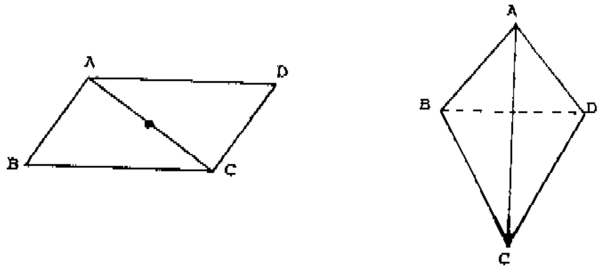

Fig 21

- A kite is any quadrilateral with at least one diagonal symmetry (a line symmetry along a diagonal).

From the above definition of a parallelogram it immediately follows that $\Delta ABC \equiv \Delta CDA$ (otherwise a half-turn symmetry would not have existed) and it therefore follows that AB = CD and DA = BC, as well as that AB//CD and DA//BC ($B\hat{A}C = D\hat{C}A$ and $D\hat{A}C = B\hat{C}A$). From the congruency we can also directly deduce that the opposite angles of a parallelogram are equal. Similarly, from the above definition of a kite and the properties of a reflection around AC it immediately follows that $BD \perp AC$, AB = AD, CB = CD and $\hat{B} = \hat{D}$.

In my opinion, strong consideration should therefore be given to alternative definitions in terms of transformations if they give rise to axiomatic-deductive structures that are more economic.

Transformations and graphs

Another useful application of transformations can be found in the study of graphs and their corresponding equations. It has often been my experience that this section is greatly neglected when doing a module on step functions. For example, asking students to plot the graphs of $y = -\text{INT}(x)$, $y = \text{INT}(-x)$ and $y = -\text{INT}(-x)$ directly after the treatment of $y = \text{INT}(x)$ they usually draw up new tables for a large number of values of x, instead of simply drawing these graphs by considering the reflections of the standard graph in the x- and y-axes.

One of the basic questions that pupils may investigate is what happens to the equations of different graphs (eg linear, parabolic, cubic, hyperbolic, trigonometric, etc) under certain transformations. Note that here is meant transformations of the graph itself, and not that of the Cartesian axes.

With the increasing availability of graphics calculators or sophisticated graphing programs on micro-computer, pupils may easily do such investigations visually and formulate general rules from that. Some examples of what happens to the parabola $y = x(x - 4)$ and its equation under certain transformations is given in Fig 22.

All straight lines are *congruent* to each other (i e can be mapped exactly on to each other by combining the isometries). It is also relatively easy to show that linearity is invariant under the affinities. Although it may seem contra-intuitive at first sight, it can also be shown that *all* parabolas are *similar* to each other (i e can be mapped exactly onto each other by some combination of the isometries and similarities). For example, in the fourth figure showing a stretch of $k = 2$ in the y-direction, the transformed parabola can be mapped exactly onto the original one by translating it upwards by 4 units and then enlarging it with $k = 2$ from the point (2; –4). A proof of the general result can be found in De Villiers (1990). Another interesting and beautiful result is that all cubic polynomials can be shown to be *point symmetrical* (eg consult De Villiers, 1989), as well as *affinely equivalent* to each other by some combination of the isome-

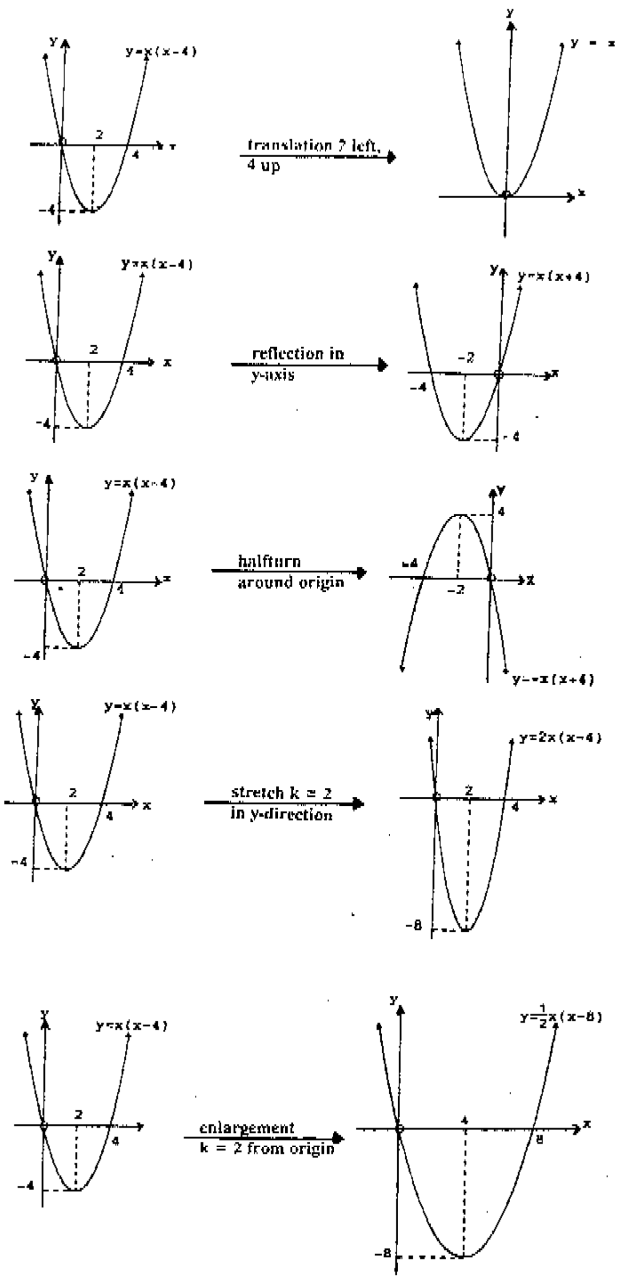

Fig 22

tries, similarities and affinities). A proof of this result can be found in De Villiers (in press (a)).

After investigating a large number of particular cases and inductively generalizing to what happens to the general equation y = f(x), it is appropriate that such general formulae are also derived from first principles using the aforementioned analytic equations. For example, from the general translation equations we simply solve for x and y in terms of x^1 and y^1 to obtain $x = x^1 - a$ and $y = y^1 - b$ and then substitute them into y = f(x) to obtain the transformed equation $y^1 = f(x^1 - a) + b$. Other examples of general formulae for y = f(x) are (dropping the primes):

Reflection in y-axis
$y = f(-x)$

Reflection in x-axis
$y = -f(x)$

Reflection in y = x
$x = f(y)$

Halfturn around origin
$y = -f(-x)$

Enlargement from origin
$y = k\,f(x/k)$

Enlargement from (p; q)
$y = k\,f[(x + p(k-1))/k] - q(k-1)$

Stretch in y-direction
$y = k\,f(x)$

Stretch in x-direction
$y = f(x/k)$

Shear in y-direction
$y = f(x) + kx$

Shear in x-direction
$y = f(x - ky)$

The equivalence between a single transformation and combinations of other transformations should also be discussed at some point in time, perhaps intuitively at first and later followed up with analytic proofs. For instance, a halfturn can be considered as the combination of two reflections, one around the y-axis and one around the x-axis. Similarly, an enlargement from the origin can be considered as the combination of two stretches in the x- and y-directions respectively. Of course, the same approach should apply to the special meaning of a reflection around $y = x$ with respect to inverse functions.

The symmetries of graphs should also generally be given more attention as they not only deal with aesthetic aspects, but can provide pupils with ample opportunity for making their own discoveries. A symmetry (line — or point symmetry) of a figure S can be defined as a rigid motion f so that for all $x \in S$, $f(x) \in S$. Presently symmetry is prescribed in Std 2 and in Murray (1988) this topic is informally treated as a prelude to the later treatment of tessellations and transformations. For example, $y = f(x)$ is symmetric around the x-axis if $y = f(x) \Leftrightarrow y = -f(x)$, around the y-axis if $y = f(x) \Leftrightarrow y = f(-x)$ and around $y = x$ if $y = f(x) \Leftrightarrow x = f(y)$. A figure is point symmetric around a certain point O if for every point P on the graph there exists a corresponding point Q also on the graph under reflection in O (see the first figure in Fig 23). Or viewed differently, if every point P on the graph can be mapped on to a point Q also on the graph by means of a halfturn around O. It therefore follows that a graph of y = f(x) is point symmetric with respect to the origin if $y = f(x) \Leftrightarrow y = -f(-x)$. Some examples of functions point symmetric around the origin are given in Fig 23. A fruitful and not very difficult further exploration could also be to examine the line and point symmetries of graphs in general, leading to the discovery of the beautiful duality between the symmetric properties of a function and those of its derivative or integral (De Villiers, in press (b)).

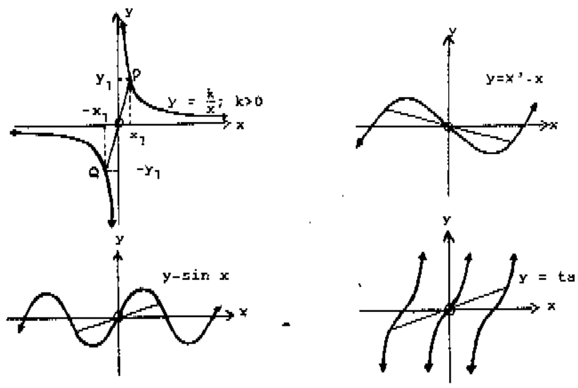

Fig 23

Using transformations in the derivation of formulae/identities

Suppose we considered a parabola of the general form $y = ax^2$ and consider its translation as shown in Fig 24. This translation involves the following transformations:

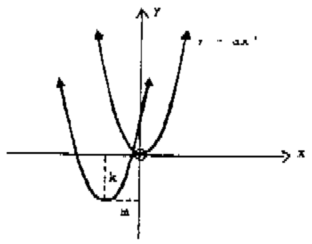

Fig 24

$y = a(x - m)^2 + k \qquad (m < 0; k < 0)$
$= ax^2 - 2max + am^2 + k$

From the graph it is immediately clear that the minimum (or maximum) can be found at $x = m$ and $y = k$. If we now compare the above quadratic equation with the standard form $y = ax^2 + bx + c$, it follows that:

1 $b = -2ma \Leftrightarrow m = -b/2a \Leftrightarrow x_{sim} = -b/2a$
2 $c = am^2 + k \Leftrightarrow k = c - am^2 \Leftrightarrow k = c - ab^2/4a^2 \Leftrightarrow y_{min/max} = -\Delta/4a$

The advantage of this approach is that it can be done directly without 'completion of the square'. Note that this derivation is valid in general since a translation to any of the four quadrants can be considered. For example, a translation to the 2nd quadrant can be considered as the following transformation: $y = a(x - m)^2 + k \quad (m < 0; k > 0)$.

Trigonometry is also a topic which lends itself nicely to the derivation of a variety of identities simply by considering appropriate transformations of the standard graphs. Consider for example the sine and cosine-graphs shown in Fig 25. Firstly, $y = \sin x$ is not symmetric around the y-axis, therefore $\sin x \neq \sin(-x)$, but $y = \cos x$ is, therefore $\cos x = \cos(-x)$. A translation of $y = \sin x$ by 180° to the right and a reflection in the y-axis gives the same graph, therefore $\sin[-(x - 180°)] = \sin x$ or rather $\sin(180° - x) = \sin x$. A translation of $y = \cos x$ by 180° to the right gives a graph which is still symmetric with respect to the y-axis and which can be mapped onto the original graph by a reflection in the x-axis, therefore $-\cos(x - 180°) = -\cos(180° - x) = \cos x$ or rather $\cos(180° - x) = -\cos x$. Similarly we can derive the identity $\sin(180° + x) = -\sin x$ by translating $y = \sin x$ to the left by 180° and reflecting it in the x-axis to obtain the original graph. We also have $\sin(180° + x) = \sin(x - 180°)$, since the same graph is obtained when using these two different transformations. A translation of $y = \sin x$ by 90° to the left gives a cosine-graph, therefore $\sin(90° + x) = \cos x$. Similarly a translation of $y = \cos x$ to the right gives a sine -graph, therefore $\cos(x - 90°) = \sin x$. But since $\cos x = \cos(-x)$ it follows that $\cos(90° - x) = \sin x$. Further experimentation with different transformations and other graphs in order to derive other familiar identities, is now left to the reader.

The stretch transformation is also frequently used in trigonometric equations. For example the transformation of $y = \sin x$ to $y = 2\sin x$ represents a stretch of $k = 2$ in the y-direction (therefore the amplitude is doubled). Similarly we have that the transformation of $y = \sin x$ to $y = \sin(x/3)$ represents a stretch of $k = 3$ in the x- direction (therefore the period increases threefold). Although this type of work in trigonometry has enjoyed a reasonable amount of attention in the past, it is unfortunate that it has not been specifically linked with the ideas and terminology of transformations.

Conclusion

This article has hopefully succeeded in its attempts to show that transformations and related concepts can fruitfully be applied in many deverse areas of school mathematics, ie that it could be seen as a golden thread. It would therefore be unfortunate if the study of transformations is abruptly stopped in Std 7 without properly utilizing their applicability in the rest of the syllabus. Although it may seem an attractive idea to completely replace traditional Euclidean geometry by transformation geometry (as in fact done in Great Britain in the School Mathematics Project and the Scottish Mathematics Group in the late 1960s and early 1970s), I am not in favour of such a radical change, but rather that the traditional curriculum should be broadened and enriched by it. Finally, transformation geometry could also provide a meaningful context for the elementary introduction of matrices at school level.

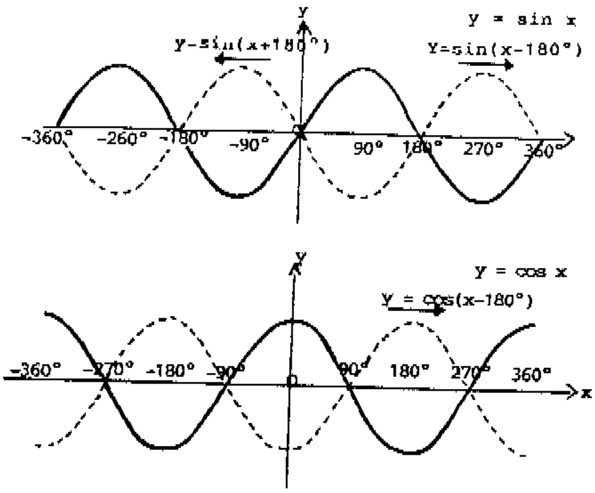

Fig 25

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