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Interesting Solutions to Two Unused Problems Proposed for the Third Round of SAMO

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The following two problems were proposed in 2002 for the Third Round of the Harmony SA Mathematics Olympiad. Unfortunately it was felt they would probably be approached and solved fairly straightforwardly by the candidates by using analytic geometry (albeit requiring some technical expertise). These problems can, however, be solved rather elegantly as follows.

Problem 1 (proposed by Michael de Villiers)
Let *ABCD* be a rectangle. Find the locus of all points *P* in the interior of *ABCD* which have the property that $\angle APB + \angle DPC = 180°$.

Solution
We first prove the following result, which appeared in the *Mathematical Digest* some years ago.

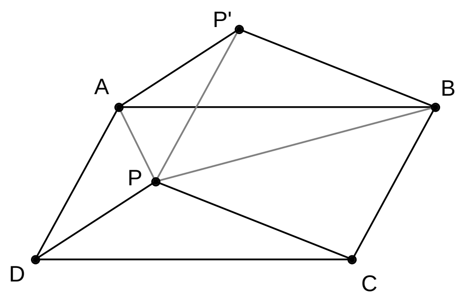

Figure 1

Lemma: Let *ABCD* be a parallelogram, and let *P* be an interior point of *ABCD*. Then $\angle APB + \angle DPC = 180° \Leftrightarrow \angle PBC = \angle PDC$.
Proof: Translate ΔDPC by vector *DA* (see Figure 1). Draw *PQ*. Then from the definition of a translation, *DA* //= *PP'* //= *BC*. Therefore, *ADPP'* and *P'PCB* are parallelograms, and *AP'* //= *DP* and *P'B* //= *PC*.

Since $180° = \angle DPC + \angle APB = \angle AP'B + \angle APB$, we have $AP'BP$ is cyclic. Hence

$\angle PBC = P'PB$ ... ($PP' // CB$)

$= \angle QAB$ ... ($AP'BP$ is cyclic)

$= \angle PDC$ ... ($\Delta P'AB \equiv \Delta PDC$).

Conversely, if we are given $\angle PDC = \angle PBC$, the same translation allows us to conclude that $\angle P'AB = \angle PDC = \angle PBC = \angle P'PB$. So $AP'BP$ is cyclic. Hence $\angle DPC + \angle APB = \angle AP'B + \angle APB = 180°$.

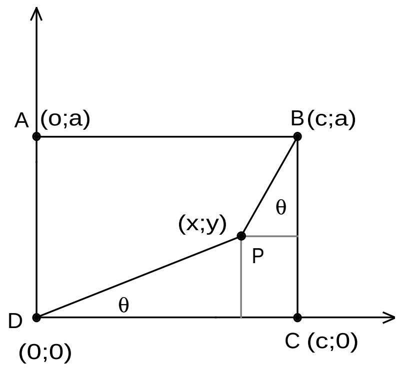

Figure 2

Solution of Problem 1

Without loss of generality, assume $a \le c$. Choose a co-ordinate frame so that *A*, *B*, *C* and *D* have co-ordinates (0; *a*), (*c*; *a*), (*c*; 0) and (0; 0) respectively (see Figure 2). Then *P* (*x*; *y*), an interior point of *ABCD* is on the required locus if and only if:

$\angle PDC = \angle PBC = \theta \Leftrightarrow \tan(\angle PDC) = \tan(\angle PBC)$ ... (angles are acute)

$$\Leftrightarrow \frac{y}{x} = \frac{c-x}{a-y} \Leftrightarrow \left(x^2 - cx\right) - \left(y^2 - ay\right) = 0$$

$$\Leftrightarrow \left(x - \frac{c}{2}\right)^2 - \left(y - \frac{a}{2}\right)^2 = \frac{c^2 - a^2}{4}.$$

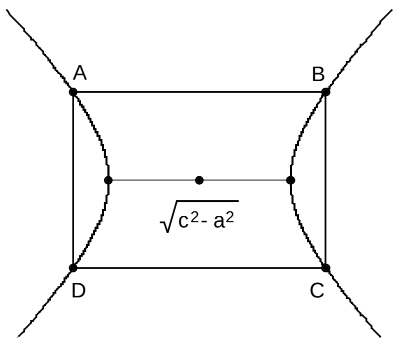

Figure 3

Hence the locus is a hyperbola centred at the centre $\left(\frac{c}{2};\frac{a}{2}\right)$, turning at $\left(\frac{c-\sqrt{c^2-a^2}}{2};\frac{a}{2}\right)$ and $\left(\frac{c+\sqrt{c^2-a^2}}{2};\frac{a}{2}\right)$, "containing" the deleted vertices of the rectangle (see Figure 3). The turning points may also be located as the points at which the half circles with sides *AB* and *DC* as diameters intersect the horizontal line through the centre of the rectangle.

Editor's Note: Problem 1 can be generalised to a parallelogram using the same approach due to the affine equivalence between a rectangle and a parallelogram and the affine invariance of a hyperbola.

Problem 2 (proposed by Sudan Hansraj)
Let Δ*ABC* be such that *AB* = *AC*. Let *D* be the midpoint of AB. Let *G* be the centroid of Δ*ADC* and let *O* be the circumcentre of ΔABC. Then line $GO \perp CD$.

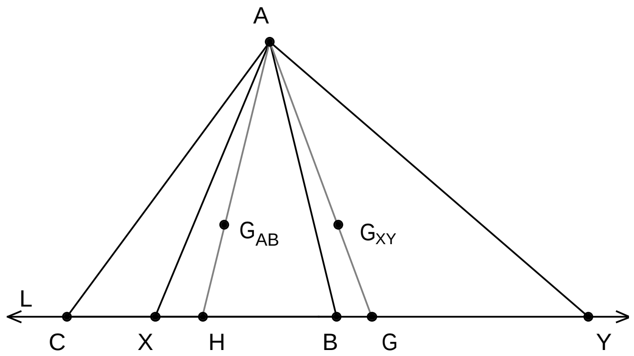

Figure 4

Lemma: Let *L* be a straight line containing points *B* and *C* with *A* a point outside line *L*. Let *X* and *Y* be any pair of points on *L* so that at least three of *X*, *Y*, *B* and *C* are distinct. Let G_{AB} and G_{xy} be the centroids of Δ*ABC* and Δ*AXY*. Then the straight line through these centroids is parallel to *BC*.

Proof: Let G_{AB} and G_{xy} produced meet *L* in *H* and *K* respectively. It is well known that $\frac{AG_{AB}}{AH}=\frac{2}{3}=\frac{AG_{XY}}{AK}$. Hence, $G_{AB}G_{XY} \,//\, BC$.

Solution of Problem 2

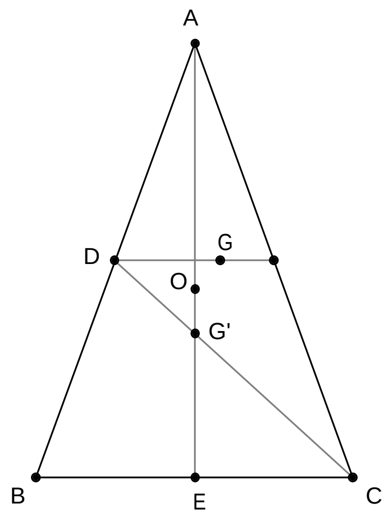

Figure 5

Let *AO* produced intersect *CD* at *G*' and *BC* at *E*. Since *AB* = *AC*, line *AOE* is the perpendicular bisector of *BC*. So *AE* and *CD* are medians of *ABC* and *G′* is the centroid of *ABC*. Thus *GG′* // *AB* by above lemma. Now *OD* ⊥ *AB* since *O* is the circumcentre; hence *OD* ⊥ *GG′*. But *G′O* ⊥ *DG* so that *O* is the orthocentre of *DGG′*, proving the assertion. (Note: *DG* produced meets the midpoint of *AC* so *DC* // *BC*, whence *G′O* ⊥ *DG*. It may happen that *O* = *G′* in which case *G′D* ⊥ *AB* and *GG′* // *AB* confirms the assertion).

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