

Interesting Solutions to Two Unused Problems Proposed for the Third Round of SAMO

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The following two problems were proposed in 2002 for the Third Round of the Harmony SA Mathematics Olympiad. Unfortunately it was felt they would probably be approached and solved fairly straightforwardly by the candidates by using analytic geometry (albeit requiring some technical expertise). These problems can, however, be solved rather elegantly as follows.

Problem 1 (proposed by Michael de Villiers)

Let $ABCD$ be a rectangle. Find the locus of all points P in the interior of $ABCD$ which have the property that $\angle APB + \angle DPC = 180^\circ$.

Solution

We first prove the following result, which appeared in the *Mathematical Digest* some years ago.

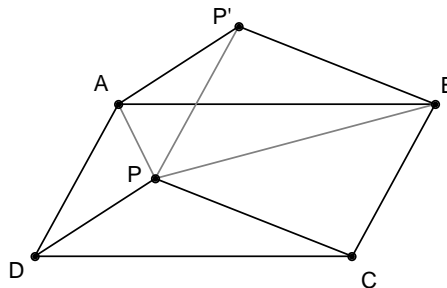


Figure 1

Lemma: Let $ABCD$ be a parallelogram, and let P be an interior point of $ABCD$. Then $\angle APB + \angle DPC = 180^\circ \Leftrightarrow \angle PBC = \angle PDC$.

Proof: Translate $\triangle DPC$ by vector DA (see Figure 1). Draw PQ . Then from the definition of a translation, $DA \parallel PP' \parallel BC$. Therefore, $ADPP'$ and $P'PCB$ are parallelograms, and $AP' \parallel DP$ and $P'B \parallel PC$.

Since $180^\circ = \angle DPC + \angle APB = \angle AP'B + \angle APB$, we have $AP'BP$ is cyclic. Hence
 $\angle PBC = \angle P'PB \dots (PP' \parallel CB)$
 $= \angle QAB \dots (AP'BP \text{ is cyclic})$
 $= \angle PDC \dots (\Delta P'AB \cong \Delta PDC).$

Conversely, if we are given $\angle PDC = \angle PBC$, the same translation allows us to conclude that $\angle P'AB = \angle PDC = \angle PBC = \angle P'PB$. So $AP'BP$ is cyclic. Hence $\angle DPC + \angle APB = \angle AP'B + \angle APB = 180^\circ$.

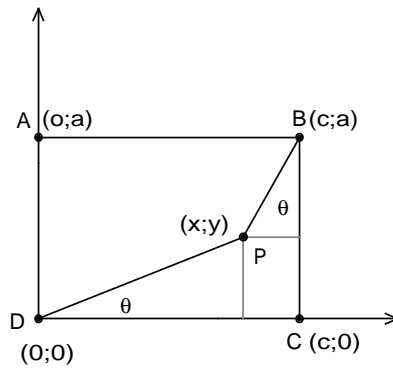


Figure 2

Solution of Problem 1

Without loss of generality, assume $a \leq c$. Choose a co-ordinate frame so that A, B, C and D have co-ordinates $(0; a), (c; a), (c; 0)$ and $(0; 0)$ respectively (see Figure 2). Then $P(x; y)$, an interior point of $ABCD$ is on the required locus if and only if:

$$\begin{aligned} \angle PDC = \angle PBC = \theta &\Leftrightarrow \tan(\angle PDC) = \tan(\angle PBC) \dots (\text{angles are acute}) \\ \Leftrightarrow \frac{y}{x} = \frac{c-x}{a-y} &\Leftrightarrow (x^2 - cx) - (y^2 - ay) = 0 \\ \Leftrightarrow \left(x - \frac{c}{2}\right)^2 - \left(y - \frac{a}{2}\right)^2 &= \frac{c^2 - a^2}{4}. \end{aligned}$$

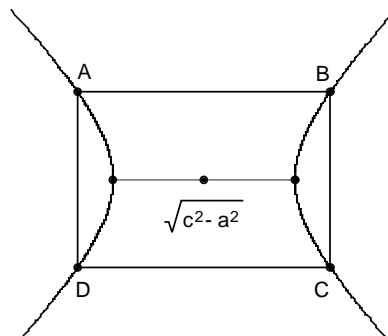


Figure 3

Hence the locus is a hyperbola centred at the centre $\left(\frac{c}{2}; \frac{a}{2}\right)$, turning at $\left(\frac{c - \sqrt{c^2 - a^2}}{2}; \frac{a}{2}\right)$ and $\left(\frac{c + \sqrt{c^2 - a^2}}{2}; \frac{a}{2}\right)$, "containing" the deleted vertices of the rectangle (see Figure 3). The turning points may also be located as the points at which the half circles with sides AB and DC as diameters intersect the horizontal line through the centre of the rectangle.

Editor's Note: Problem 1 can be generalised to a parallelogram using the same approach due to the affine equivalence between a rectangle and a parallelogram and the affine invariance of a hyperbola.

Problem 2 (proposed by Sudan Hansraj)

Let $\triangle ABC$ be such that $AB = AC$. Let D be the midpoint of AB . Let G be the centroid of $\triangle ADC$ and let O be the circumcentre of $\triangle ABC$. Then line $GO \perp CD$.

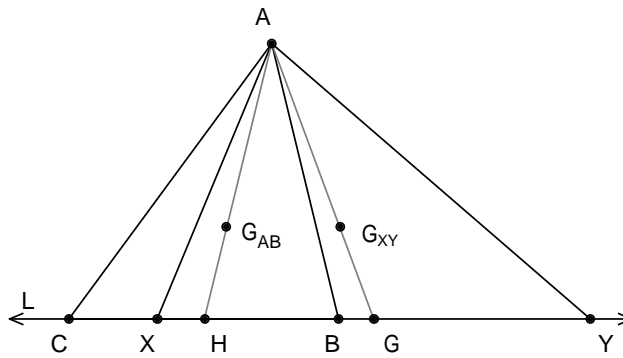


Figure 4

Lemma: Let L be a straight line containing points B and C with A a point outside line L . Let X and Y be any pair of points on L so that at least three of X, Y, B and C are distinct. Let G_{AB} and G_{xy} be the centroids of $\triangle ABC$ and $\triangle AXY$. Then the straight line through these centroids is parallel to BC .

Proof: Let G_{AB} and G_{xy} produced meet L in H and K respectively. It is well known that $\frac{AG_{AB}}{AH} = \frac{2}{3} = \frac{AG_{xy}}{AK}$. Hence, $G_{AB}G_{xy} \parallel BC$.

Solution of Problem 2

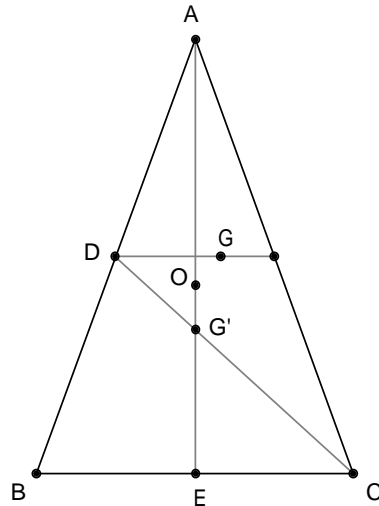


Figure 5

Let AO produced intersect CD at G' and BC at E . Since $AB = AC$, line AOE is the perpendicular bisector of BC . So AE and CD are medians of ABC and G' is the centroid of ABC . Thus $GG' \parallel AB$ by above lemma. Now $OD \perp AB$ since O is the circumcentre; hence $OD \perp GG'$. But $G'O \perp DG$ so that O is the orthocentre of DGG' , proving the assertion. (Note: DG produced meets the midpoint of AC so $DC \parallel BC$, whence $G'O \perp DG$. It may happen that $O = G'$ in which case $G'D \perp AB$ and $GG' \parallel AB$ confirms the assertion).

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for past papers and links to many problem solving sites!

Solutions will be posted immediately

after this year's papers!