Pythagoras, 33, April 1994, pp.33 – 36

A mathematical look at "voting power"

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On 27 – 29 April 1994 a much-awaited historic event is scheduled to take place in South Africa, namely, the first fully democratic election. At this point in time, it might therefore be of particular interest and importance for mathematics teachers and pupils to look at the mathematical aspects of voting concepts like "*voting power*". In a modest way, this could positively contribute to the general voter education of our nation.

Another purpose of this article is to demonstrate the application of mathematical techniques to an aspect of political science, and thereby challenging the stereotype that mathematics is of value only in certain applied sciences like Physics, Chemistry, Computer Science, etc. It is aimed at the junior high school level and mainly involves elementary mathematical concepts such as percentages, counting techniques, etc.

What is voting power?

Most definitions of "*voting power*" by political scientists take the view point that the greater the proportion of votes an actor (e.g. a party) controls, the greater is that actor's power. Phrased differently, one's voting power is directly proportional to the number of votes one controls. But is this really the case?

Let's look at a specific example. Suppose in an election Parties A, B and C respectively win 200 (44,4%), 150 (33,3%) and 100 (22,2%) seats in a 450-seat assembly. Intuitively, it would appear that Party A does have more power than the other two; however, is that really true?

The Banzhaf method

The Banzhaf Index has been accepted by the New York State Court of Appeals as a basis for assigning weights to representatives on that state's County Boards of Supervisors. This method examines all the possible outcomes of voting by the actors in a voting body.

For simplicity's sake, we shall assume here that all the party representatives of any specific party always vote the same, e.g. in a block. (In practice this doesn't necessarily happen).

In general, there are 2^n possible yes or no outcomes of a voting by n parties (groups of voters). In the above three-party example, we have $2^3 = 8$ possible outcomes under the *simple-majority* rule (votes ≥ 225) as shown in Table 1.

We count the total number of yes votes to determine whether the issue would pass (votes that equal or exceed the required minimum) or would fail. Examine each outcome. If the issue passes, we then look at each party that voted yes. If *any one party at a time* were to change to a no vote, would the issue then fail? If it would, circle that party, which is called **pivotal** (or critical), and continue to check all the parties for each outcome. (Note: More than one party or no party may be pivotal on one

Parties				
A(200)	B(150)	C(100)	Total Votes	Result
Yes	Yes	Yes	450	Pass
(Yes)	No	(Yes)	300	Pass
(Yes)	(Yes)	No	350	Pass
Yes	(No)	(No)	200	Fail
No	(Yes)	(Yes)	250	Pass
(No)	(No)	Yes	100	Fail
(No)	Yes	(No)	150	Fail
No	No	No	0	Fail

Table 1

line). Similarly, if the issue fails, then we check each party that voted no. If any one of these parties changes to a yes vote, will the issue then pass? If it will, that party is pivotal and circle it.

For example, when all three parties vote yes, the issue passes with 450 votes. If any *one* party changes its votes to no, the issue will still pass; therefore, in the first line no party is pivotal. In the second line, Parties A and C voted yes for a total of 300 votes, so the issue passes. However, if either party changes to a no vote, the issue will fail. As a result, both Parties A and B are pivotal.

The voting power of each party is now calculated by dividing the number of times that party is pivotal by the total number of pivotal cases. Thus, the Banzhaf Index for parties A, B and C is $(4/12, 4/12, 4/12) = (1/3, 1/3, 1/3)$. Furthermore, the voting power of each *individual* member of the parties can be calculated as follows:

Party A = $1/200 \times 1/3 = 1/600$
Party B = $1/150 \times 1/3 = 1/450$
Party C = $1/100 \times 1/3 = 1/300$

As one can see, Party A does *not* possess more power. Rather, all three parties possess the same power because no *one* party can win, but any *two* parties can. Furthermore, when one examines the voting power of the individual members, the members of Party C, not Party A, possess the most power.

Exercise 1

(a) What happens to the voting power of the respective parties in the example above if the voting rule is changed to the $2/3$-majority rule (votes $\geq$ 300)? Before you do the actual calculation, which party or parties do you intuitively think will benefit most from an increase in the voting rule?

(b) What is the respective voting power of three Parties A, B and C under the simple-majority rule if they respectively have 200(57%), 100(29%) and 50(14%) seats in a 350-seat parliament or house of assembly?

The paradox of new members

Conventional political wisdom suggests that a stratagem for diluting the power of a dominant member (or coalition) in a voting body is to increase the size of the body. But is that really the case?

Suppose a new party with 50 seats, say Party D, is added to our first example, thus increasing the 450-seat parliament to 500 seats. How does this increase in the size of the body affect the respective voting power of the original parties, given a voting rule of simple majority (votes $\geq$ 250)?

Consider Table 2 which shows all possible outcomes of a voting by the representatives of the four parties ($2^4 = 16$). The pivotal cases have already been circled. We now have the Banzhaf Index for the parties A, B, C and D as $(10/24, 6/24, 6/24, 2/24) = (5/12, 1/4, 1/4, 1/12)$.

While the voting power of Parties B and C decreases in the enlarged voting body, Party A surprisingly *in-*

Parties					
A (200)	B(150)	C(100)	D(50)	Total Votes	Result
Yes	Yes	Yes	Yes	500	Pass
Yes	Yes	No	Yes	400	Pass
Yes	Yes	Yes	No	450	Pass
Yes	Yes	No	No	350	Pass
Yes	No	Yes	Yes	350	Pass
Yes	No	No	Yes	250	Pass
Yes	No	Yes	No	300	Pass
Yes	No	No	No	200	Fail
No	Yes	Yes	Yes	300	Pass
No	Yes	No	Yes	200	Fail
No	Yes	Yes	No	250	Pass
No	Yes	No	No	150	Fail
No	No	Yes	Yes	150	Fail
No	No	No	Yes	50	Fail
No	No	Yes	No	100	Fail
No	No	No	No	0	Fail

Table 2

creases its power in the body! Specifically, its power increases from 1/3 = 0,33 to 1/12 = 0,42, despite the fact that its relative proportion of votes decreases from 200/450 = 0,44 to 200/500 = 0,40. This simultaneous *decrease* in a party's proportion of votes in an enlarged voting body, and its *increase* in voting power, certainly seems paradoxical, especially since the new party is not entirely powerless and – by its presence – deprives the other parties together of some voting power.

Interestingly, Brams (1976) shows that the probability of this paradox occuring when one party is added to a voting body (with between 2 and 7 parties) always exceed 0,45; a fairly high probability. He also gives interesting empirical examples of this paradox and its consequences in American politics and the EEC.

Exercise 2

(a) What happens to the voting power of the four parties above if the voting rule is changed to the 2/3-majority rule (votes ≥ 333)?
Before you do the actual calculation, which party or parties do you intuitively think will in this case benefit most from an increase in the voting rule?

The paradox of quarrelling members

A traditional view of power is that the less conflicts an actor (e.g. a party) has with other actors, the greater is that actor's power. Suppose that two parties in a voting body are involved in a quarrel and refuse to work together in forming coalitions. (In other words, suppose they refuse to work together *in support* of a motion; i.e. refuse to simultaneously vote "*yes*"). Intuitively one would expect that they would only succeed in hurting each other; "*cutting the nose to spite the face*". But is this necessarily the case?

Consider again the problem given in Exercise 1(a) where we found a Banzhaf index of (3/5, 1/5, 1/5). The pivotal cases are shown in Table 3. Suppose the two quarrelling parties are B and C. Delete the first and fifth lines since these are precluded by the quarrelling restriction that they do not work together in support of a motion. Then the Banzhaf index becomes (4/8, 2/8, 2/8) = (1/2, 1/4, 1/4). Surprisingly, the voting power of parties B and C has increased from 1/5 = 0,2 if they do not quarrel to 1/4 = 0,25 if they quarrel!

As Brams (1976:190) points out:

"... *therefore* ... *power considerations – independently of ideological considerations – may inspire conflicts among members of a voting body simply because such conflicts enhance the quarrelling members'* ... *voting power.*"

Conclusion

Of course, one might argue that the aforementioned paradoxes are "*aberrant*", merely artificial creations of the Banzhaf index. However, it should be pointed out that exactly the same paradoxes arise with other measures of voting power, e.g. the Shapley-Shubik and Coleman indices (see Swetz & Hartzler, 1991:60-64; Brams, 1976:176-190).

In fact, Brams (1976:192) takes the viewpoint that it has been a limitation in traditional political thinking and models, not an aberration in the phenomenon, to simply equate power with size and the lack of conflict. It there-

Parties				
A(200)	B(150)	C(100)	Total Votes	Result
~~(Yes)~~	~~Yes~~	~~Yes~~	~~450~~	~~Pass~~
(Yes)	No	(Yes)	300	Pass
(Yes)	(Yes)	No	350	Pass
Yes	(No)	(No)	200	Fail
~~(No)~~	~~Yes~~	~~Yes~~	~~250~~	~~Fail~~
(No)	No	Yes	100	Fail
(No)	Yes	No	150	Fail
No	No	No	0	Fail

Table 3

fore seems preferable to view these paradoxes as reflecting subtle aspects of voting power whose existence would have been difficult to ascertain in the absence of precise quantitative concepts.

Readers may also wish to consult Brams (1976) for an interesting discussion of the paradox of a chairperson's position in a voting body, who in some cases may be at a distinct disadvantage in relation to other members, despite his/her extra tie-breaking vote. (Also see Steiner, 1968).

Solutions

Exercise 1

(a) Intuitively, one may expect that the increased voting rule would decrease the voting power of the largest party, since one could argue that it would become more dependant on other parties to form winning coalitions.
However, the Banzhaf index is (6/10, 2/10, 2/10) = (3/5, 1/5, 1/5). (See Table 3). In other words, the largest party's voting power almost *doubled* from 1/3 = 0,33 to 1/5 = 0,60, while the two smaller parties voting power decreased from 1/3 = 0,33 to 1/5 = 0,20! It therefore seems reasonable to assume that in a situation like that shown in Table 1, Party A would strongly favour an increase in the voting rule. (Assuming of course they are aware of the Banzhaf Index).

(b) In this case, the Banzhaf Index is (1, 0, 0). Since Parties B and C together only control 150(43%) of the votes, their votes are irrelevant to the selection of outcomes. Such powerless members are usually called *dummies*. Party A, on the other hand, is called a *dictator*, since its votes by themselves are sufficient to determine the outcome, and only coalitions of which it is a member.
Note that in the situation in Table 1, a decrease of the voting rule to 40% (votes ≥ 180) would result in party A becoming a dictator.

Exercise 2

(a) The Banzhaf Index is (10/20, 6/20, 2/20, 2/20) = (1/2, 3/10, 1/10, 1/10). Interestingly, although the voting power of the two larger parties both increased as may be expected from our previous example, the voting power of party D also slightly increased from 1/12 = 0,08 to 1/10 = 0,1. However, Party C had a huge decrease in voting power from 1/4 = 0,25 to 1/10 = 0,1.

References

Brams, S.J. (1976). **Paradoxes in politics.** New York: The Free Press.

Swetz, F. & Hartzler, J.S. (1991). **Mathematical modelling in the secondary school curriculum.** Reston, VA: NCTM.

Steiner, H.G. (1968). Examples of exercises in mathematization on the secondary school level. **Educational Studies in Mathematics**, 1:181-201.

Voting power revisited

In the previous **Pythagoras** we looked at the Banzhaf index as a means of measuring the voting power of actors in a voting body. Now that votes have been counted after the April elections and seats allocated in the national and provincial bodies, it may be interesting for mathematics teachers and their pupils to look at the power of the various parties in these bodies. (In fact, as will be shown such analyses could uniform and direct policy decisions and strategic moves by certain parties).

The Banzhaf Indices (B.I.) of the voting power of the various political parties at national level is given in Table 1. The Banzhaf indices are given for simple majority (votes > 50%), as well as two-thirds majority (votes > 66.7%). The voting power of the individual members of each party is given in brackets. Also note that the percentage after the number of seats of a party is the percentage of seats at national level and not necessarily equal to the percentage of the vote obtained in the election.

It should immediately be clear that for decision making by simple majority the ANC will have all the power (if we assume that all ANC members always vote the same), and that all the other parties are completely powerless to determine the outcome of any vote.(In the mathematics of voting theory a member of a voting body who controls all the power is called a *dictator*, while powerless members are called *dummies*). However, for two-thirds majority it is interesting to see that the individual voting power of a minority party, namely the Freedom Front, is the highest at 0.0065. Furthermore, although the National Party has the second most seats, each individual member has the smallest ability to determine the outcome of a vote.

Interestingly, although the NP has almost twice as many seats as the IFP in the National Assembly, their voting power is exactly the same at 0.129. It is also interesting to see as shown in the second last column that the formation of a coalition between say the DP and FF does not decrease the voting power of the ANC as one might intuitively expect, but increases it from 0.624 to 0.70. Furthermore, the formation of this coalition would give them the same power as the NP and IFP, and would result in disempowering the PAC and ACDP entirely.

Far more significantly, we see as shown in the last column that a (minimum) coalition of the NP, IFP and FF would result in them obtaining exactly the same Banzhaf Index of 0.5 as the ANC! In other words, such a coalition would have exactly the same power to determine the outcome of a vote as the ANC, even though the ANC has almost twice their number of seats. Under such conditions, it would be advisable for the smaller, now powerless parties, to join one of these groupings. (It should be noted that 267 as the minimum for the two thirds passing of a vote is extremely critical here. If 266 is the minimum for the passing of a vote, then the DP, PAC and ACDP would have some voting power as they can together combine with the ANC to ensure the passing of a vote. In fact, the Banzhaf Indices would then be: ANC = 0.44; NP/IFP/FF = 0.39 and DP = PAC = ACDP = 0.06. If 266 is chosen as the minimum for the passing of a vote, at least one of the smaller parties would then also have to join the coalition of the NP/IFP/FF for the coalition to again have the same power as the ANC at 0.5).

Finally, teachers and their pupils may also wish to determine the Banzhaf indices in the different provincial assemblies, and consider the effect of different coalitions.

National

	Seat	B.I. (> 200)	B.I. (> 266)	B.I. (> 266)	B.I. (> 266)
ANC	252 (63.0%)	1(0.004)	0.624 (0.0025)	0.70 (0.003)	0.50 (0.002)
NP	82 (20.5%)	0	0.129 (0.0016)	0.10 (0.001)	0.50 (0.004) (NP, IFP, FF)
IFP	43 (10.8%)	0	0.129 (0.0030)	0.10 (0.002)	
FF	9 (2.2%)	0	0.059 (0.0065)		
DP	7 (1.8%)	0	0.035 (0.0050)	0.10 (0.006) (FF, DP)	0
PAC	5 (1.3%)	0	0.012 (0.0024)	0	0
ACDP	2 (0.5%)	0	0.012 (0.0059)	0	0

Table 1